

# **Calculus in upper secondary and beginning university mathematics**

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Edited by John Monaghan, Elena Nardi and Tommy Dreyfus

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# Introduction

## Calculus in upper secondary and beginning university mathematics

The genesis of the conference were discussions between us on what was being taught – and what could be taught – under the name ‘calculus’ in schools, colleges and universities in our countries. Whilst we have nothing against the differentiation and integration techniques of calculus, we saw and see calculus as much more than a set of techniques; at its heart is covariational change, rates of change and accumulations, which can be overlooked by foci on techniques. The idea for the conference came when we realised that MatRIC ([www.matric.no](http://www.matric.no)), the Norwegian centre for excellence in university mathematics teaching, might support a conference on calculus and allow a bigger audience for our discussions; and MatRIC agreed to support us.

The conference had four themes:

1. The school-university transition with a focus on calculus
2. The fundamental theorem of calculus
3. The use of digital technology in calculus
4. Calculus, and its teaching and learning, for various disciplines

We comment on these themes but first note that calculus at school and calculus at university are not, at a global level, clearly differentiated areas or approaches to calculus: what is taught at school in one country may be taught at university in another country; university calculus courses, especially for non-mathematics majors, often revisit school calculus content. The latter, revisiting school calculus content at university, is one reason why the school-university transition, our first theme, is an important focus. Another reason why this focus is important is the jump from what might be called informal (or elementary) calculus at school to formal calculus (or analysis) at university, the latter involving  $\varepsilon$ – $\delta$  definitions and proofs. Not all students will experience this jump but mathematics majors will – and this jump to rigour is a noted area of difficulty for many students. The fourth theme, on teaching and learning calculus in various disciplines, is related to the first theme in as much as this teaching and learning occurs, almost exclusively, at the university level, to students who have come from school. It is an important theme in itself: what are appropriate calculus courses for economists, for engineers, etc. and how might these differ (in content and approach) to calculus courses for mathematics majors? This is also a developing area in terms of practical approaches and mathematics education research.

The third theme, on the use of digital technology in calculus, exists because (i) mathematical software has the potential to restructure what and how calculus is taught and learnt and (ii) there are many initiatives that essentially incorporate digital technology in the teaching and learning of calculus.

The second theme is the only theme to focus on a specific part of calculus, the fundamental theorem. It is there for many reasons. It was the main focus in our initial discussions on calculus. It is a beautiful and important part of calculus. It is quite often covered scantily and/or badly – including the maximal bad approach, the statement “integration is the reverse of differentiation”. There is at least one

approach to teaching it (as part of a full calculus course) that we all respect – Pat Thompson’s Project DIRACC.

The themes were built into the plenaries: Rolf Biehler was briefed to address the first and fourth theme and Pat Thompson was briefed to address the middle two themes in their plenary addresses and papers. The themes also partially structured the contributed papers in as much as invitations to present a paper went out with the proviso that the paper should address at least one of the four themes.

We now provide an overview of the plenary lectures, plenary panel and contributed papers presented but first repeat that the conference was conceived, from outset to completion, as a discussion conference: the conference was more than the sum of the plenary contributions and the presented papers!

The opening plenary was by David Tall. David was not well enough to attend the conference and provided a video which you can watch on <https://matric-calculus.sciencesconf.org/>. David has, almost certainly, written more papers on the teaching and learning of calculus than anyone else in the world, from 1975 to today. Amongst many other things, he is noted for his software (SuperGraph), the concept of local straightness and his Three Worlds approach. His plenary paper, *The evolution of Calculus: A personal experience 1956-2019*, addresses issues that touch on the first three conference themes.

Rolf Biehler has invested much of his academic thought and effort in the past decade to issues concerned with beginning university studies, in which calculus or analysis play a central role. The main focus of his plenary, *The transition from calculus to analysis – conceptual analyses and supporting steps for students*, is on the transition problem and concerns the move from school calculus to university analysis and the culture of mathematics that underlies these courses. Students need help in this cultural move and his paper outlines and evaluates measures to support students in this enculturation.

It is 25 years since Pat Thompson went public on his ideas for dynamic student experiences leading to the fundamental theorem of calculus in a paper in *Educational Studies in Mathematics*. Since then, he has developed a complex and highly respected calculus curriculum based in quantitative reasoning, differentials as variables, rate of change and accumulation in which student and instructor use of digital resources as didactic objects, not as mere tools for illustration, is essential. He outlines practical and theoretical issues related to this curriculum in his plenary *Making the Fundamental Theorem of Calculus fundamental to students’ Calculus*.

The plenary panel, entitled *From Newton’s first to second law: How can curriculum, pedagogy and assessment celebrate a more dynamic experience of calculus?*, had contributions from Alejandro S. González-Martín, Vilma Mesa and John Monaghan who set out from acknowledging that, while digital and other resources provide the capacity to create and celebrate dynamic ways in experiences Calculus, there are numerous institutional and other challenges that may impede embracing this capacity. The three panelists addressed those first through sharing their experiences in the study and design of curriculum and assessment materials for calculus. They then mapped out one possible way of fostering change: designing tasks—for classroom activity as well as assessment—that convey

important meanings of calculus, are accessible, celebrate its dynamism, and are tailored to the needs of students in various disciplines who will soon enter diverse worlds of work.

There are 27 contributed papers. This number is not accidental: we wanted the conference to enable each paper to be discussed in depth and with regard to similar papers; 3 of the conference days had 3 parallel ‘paper sessions’ with 3 papers in each session and  $3 \times 3 \times 3 = 27$ . The contributed papers were grouped in 3s, according to their focus. You can see the grouping by going to the conference programme on the website: for example, the papers by Feudel, González-Martín and Viirman in strand C on the Thursday of the conference all concern calculus for students not majoring in mathematics. We are pleased with the variety and quality of the papers. Readers should be aware that the authors had the challenge of presenting complex ideas in just four pages – not an easy task!

The conference is over but its brief to encourage discussion on what and how is being – and could be – taught under the name ‘calculus’ continues. Special issues of the *International Journal for Research in Undergraduate Mathematics Education* and *Teaching Mathematics and its Applications*, based on papers and collaborations arising from the conference, are planned. At least one potential research project, *Calculus in upper secondary school and in teacher education – an overview of current practices*, is in its inception and follow up conferences, in which we aspire to attract participation by the many colleagues from around the world not present in this first conference, are under discussion. The conference website (<https://matric-calculus.sciencesconf.org>) will, for the time being, keep people abreast of developments in these areas.

John Monaghan, University in Agder, Norway and University of Leeds, UK;  
[john.monaghan@uia.no](mailto:john.monaghan@uia.no)

Elena Nardi, University of East Anglia, UK; [e.nardi@uea.ac.uk](mailto:e.nardi@uea.ac.uk)

Tommy Dreyfus, Tel Aviv University, Israel; [TommyD@tauex.tau.ac.il](mailto:TommyD@tauex.tau.ac.il)

## Plenary paper

# The transition from Calculus and to Analysis – Conceptual analyses and supporting steps for students

Rolf Biehler

University of Paderborn, Germany; [biehler@math.upb.de](mailto:biehler@math.upb.de)

### Abstract

*The paper starts with discussing the transition problem by characterizing the type of mathematics that is characteristic of school calculus. General types of measures for supporting students in the transition phase will be briefly reviewed. An essential aspect of the transition problem is the new culture of mathematics that is underlying Analysis courses. Arguments for making this change in culture more explicit and some concrete suggestions will be provided. The paper discusses examples from two empirical studies to support this analysis. In the first study, some results of an Analysis 1 final examination course in the first semester are analyzed. One task of the examination is taken, where school-mathematical solution strategies conflict with university-based norms. A second example is taken from a design-based research study, where a workshop was designed for supporting the guided reinvention of the concept of convergence of a sequence before formally introducing it in the lecture. One lesson learned of this second study is to be much more explicit about which conditions mathematical definitions at university have to fulfil, which seldom is a topic of explicit instruction.*

### Introduction

Students in Germany obligatorily learn Calculus in the last years of the college-bound schools leading to the Abitur (Gymnasium) and usually take courses on Analysis in their first year of studies if they have chosen a subject that has a mathematical component. These Analysis courses vary depending on whether the students have entered a programme for mathematics majors or, for instance, for future engineers or economists. In general, the university courses on Analysis in Germany are similar in mathematical style to the Real Analysis courses in the US, and the calculus courses at colleges in the US are somewhat "in-between" the school calculus courses in Germany and the University Analysis courses. In other words, the transition from Calculus to Analysis in Germany coincides with the secondary-tertiary transition.

### Calculus at the school level: sources of transition problems

The relatively new German national standards for the last three years before the Abitur include standards for teaching calculus (KMK, 2012). However, it is certainly not wrong to say that these standards do not focus on easing the transition to Analysis at the university level. The transition is relevant for a substantial part of students, namely those who study STEM subjects or economy. However, these students are not the majority. Calculus teaching has to provide an excellent education for a broader audience, where aspects of general education (Allgemeinbildung) have to be taken into account (Biehler, 2019). A significant concern of the mathematics education community is that students do not just develop calculation skills at the school level but also the conceptual understanding of the basic concepts. The notion of "Grundvorstellungen" ("basic mental models" is an approximate

translation) is used to characterize this understanding. This notion is related to the notions of mental models and concept image (Greefrath, Oldenburg, Siller, Ulm, & Weigand, 2016). Derivative should be understood as the local rate of change, the slope of the tangent, or the best linear approximation. The cumulation aspect of the integral including reconstructing the “stock function” from the rate of change function should be important conceptual ideas, besides understanding the definite integral as a measure for the oriented area under a function graph. These “Grundvorstellungen” are particularly necessary for the effective use of integrals and derivatives in modelling contexts. The theoretical aspects of calculus lie not in the centre of a typical course.

Recently, there have been efforts to offer non-compulsory additional courses already at the school level to students who are interested in entering study programs in STEM subjects. This trend is in line with recent recommendation of the German associations of mathematics, didactics of mathematics and STEM teachers<sup>1</sup> who also recommend measures for the first semester of university studies such as pre-university bridging courses and renewed curricula in the first study year. These recommendations are backed up by a recent Delphi study in which more than 800 German mathematics university teachers were asked, what they consider as necessary prerequisites for students entering a STEM university program (Neumann, Heinze, & Pigge, 2017; Pigge, Neumann, & Heinze, 2017).

A recent book that stems from such a course offered at school level is the book by Proß and Imkamp (2018), which – in large parts – focusses on the transition between Calculus and Analysis. Similar content is also covered in university-based pre-university bridging courses such as the studiVEMINT course (<http://go.upb.de/studivemint>), where the author of this paper is a co-developer of. I will take the book by Proß and Imkamp (2018) and the studiVEMINT course as an example of a view of what topics and competencies are currently missing in school mathematics for easing the transition. The courses improve the “technical basis” for Analysis at the university level in that covers algebraic techniques such as inequalities including those with the absolute value function, and equations with powers, roots and logarithms and of course practices to work with algebraic terms (including those with fractions). However, this is not a new type of mathematics. These topics "disappeared" from school mathematics in recent years.

In contrast to textbooks at the school level, the presented rules are mathematically explained and justified if not proven. In the terminology of ATD (the Anthropological Theory of Didactics) the technological block is expanded. However, different from university courses (Winsløw & Grønbæk, 2014), the new technology in the sense of ATD does not become a practice on a higher level: students are supposed to fluently calculate but not to prove theorems about roots and logarithms.

A different approach elaborates the function concept through emphasizing related concepts such as injectivity, surjectivity, composition of functions and inverse functions are treated as well as domain and co-domain emphasizing functions as mappings. This perspective on functions is different from the current perspective in school mathematics. We also find a chapter on sequences and limits, including its formal definitions. The topic “sequences and limits” has practically disappeared from

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<sup>1</sup> [http://mathematik-schule-hochschule.de/images/Massnahmenkatalog\\_DMV\\_GDM\\_MNU.pdf](http://mathematik-schule-hochschule.de/images/Massnahmenkatalog_DMV_GDM_MNU.pdf)

the secondary curriculum. Often, limits are introduced the first time when the derivative concept is introduced, but not as a separate chapter. If school textbooks have separate chapters on sequences and limits, these chapters are often non-compulsory, and sequences and limits are – if at all - introduced for modelling discrete processes of real-life situations but not as fundamental theoretical means for the conceptual development of calculus. The above book also contains a chapter on continuity and a section on differentiability. Both concepts do not play a significant role in secondary schools. Continuity usually has no separate chapter in a school textbook. Continuity is sometimes mentioned as a graphical property of functions ("one can draw the graph by a pencil without jumps and holes"). As the fundamental theorem of calculus (FTC) is a compulsory topic in secondary mathematics – all students should know a visual proof of that theorem according to the standards. The intuitive concept of continuity is mentioned as an assumption, but it is rarely pointed out, why this assumption is essential, and it is often quickly forgotten by students because the most important "meaning" of the FTC at school level is the justification for determining values of definite integrals by using primitive functions. This meaning is represented in the practical block surrounding the FTC. Proß and Imkamp (2018) and the studiVEMINT bridging course also discuss differentiability and non-differentiability of a function as a conceptual prerequisite for defining the derivative, whereas most school textbooks do not question the existence of a derivative, which is correct for nearly all the functions in a school calculus course. There are some notable exceptions, however, which are downplayed in most school textbooks. The integral is often introduced by starting with step functions in a modelling context, where the oriented area under a step function can be given a meaning in the context. For instance, if the step function can be interpreted as velocity, the area function stands for the covered distance. The area function depending on a variable  $x$  can be calculated using elementary geometry. The resulting area function has points of non-differentiability where the step function changes its value (where it has points of discontinuity).

These differences are discussed in teaching material for courses in didactics of calculus (Biehler, 2018) and based on an analysis of selected textbooks. A systematic textbook analysis would undoubtedly confirm the compartmentalization structure of school calculus focussing on separated practical blocks similar to what analyses of other topics from the perspective of ATD have revealed (González-Martín, Giraldo, & Souto, 2013). This structure of school calculus is fundamentally different from university analysis. An even deeper lying epistemological difference was brought into focus based on textbook analyses by Witzke and colleagues (Witzke, 2014; Witzke & Spies, 2016) who characterize school calculus as an empirical theory of graphically represented function graphs. This characterization is related to the fundamentally different role graphs play in school Calculus and in university Analysis (Weber & Mejía-Ramos, 2019).

### **Transition problems in the first study year**

In general, at the university level in Germany, we can observe three types of measures for supporting students in the transitional phase. In the WiGeMath project, we are collaborating with 14 German universities with regard to analysing and evaluating support measures. We analyse the most common types of measures: pre-university bridging courses, so-called bridging lectures for the first study year that are especially designed for easing the transition, and measures that accompany traditional lectures such as the creation of mathematics support centres, non-compulsory additional tutorials, or

supportive e-learning material (Kuklinski et al., 2018; Liebendörfer et al., 2017). What is not so common in Germany is a fundamental content redesign of the standard introductory lectures, i.e. Analysis and Linear Algebra.

A wide-spread measure in German universities is to offer pre-mathematical bridging courses for future students to support the transition process. Universities offer these courses in the months between attending school and starting at university, and they last between two and six weeks. Calculus and Analysis play a role in most of these courses, goals and orientations of these courses vary considerably (Biehler & Hochmuth, 2017; Biehler et al., 2018). Some courses repeat school mathematics, and others introduce mathematical concepts at a level of rigour similar to later university mathematics. The latter type of bridging course aims at preparing students for the upcoming level of rigour and a changed mathematical practice or praxeology. Among others, they already practice the new role for formal definitions and theorems, and the new role of arguments based on graphical representations, which are no longer accepted as valid arguments but only as preparatory "heuristic" ones. Some courses make the transition into a new practice with different features explicit instead of just working on one of the levels. In these types of courses, the conscious change of the expectations and the mathematical belief systems of the students stand in the focus while taking their previous knowledge and orientations into account.

Felix Klein is often quoted speaking about the "double discontinuity" of the transitions that mathematics teachers have to face: school mathematics  $\rightarrow$  university mathematics  $\rightarrow$  school mathematics (Winsløw & Grønbaek, 2014). We are concerned with the first transition in this paper, but for the second transition in calculus, see Wasserman, Weber, Fukawa-Connelly, and McGuffey (2019).

The new praxeology of university mathematics is – as a rule – not an explicit topic of teaching in the standard first semester courses such as Analysis. The new style of defining, proving and theory development is practised and communicated through examples in the lecture. A decisive role for the enculturation has the feedback that students receive to their written homework, either as written comments to their submitted work or orally in weekly tutorial sessions, where norms for adequate proofs and written argumentations are discussed at least implicitly. Part of the students does not understand and master this largely implicit enculturation. In a joint project with Leander Kempen (Kempen & Biehler, 2019a, 2019b), we newly designed a first-semester course called "Introduction into the culture of mathematics". The accompanying research followed a design-based research paradigm, and we re-designed the course three times based on the previous research results. One of the significant changes was that we became more and more explicit about the question of what counts as proof and what is an adequate use of formal mathematical language. We used the language of socio-mathematical norms (Yackel & Cobb, 1996) for the analysis and redesign of the course. A redesign of an Analysis 1 course may profit from this approach.

In the following, examples from two empirical studies done at the khdm (competence centre for research in university mathematics education, [www.khdm.de](http://www.khdm.de)) are presented to support this analysis. In the first study, the results of an Analysis 1 final examination at the end of the first semester are analyzed. One task of the examination is taken, where school-mathematical solution strategies



conflict with university-based norms. A second example is taken from a design-based research study, where a workshop was designed for supporting the guided reinvention of the concept of convergence of a sequence before formally introducing it in the lecture. One lesson learned from this second study is to be much more explicit about which conditions mathematical definitions at university have to fulfil, which seldom is a topic of explicit instruction.

### Conflicts between school mathematics and university mathematics education: An example from a final Analysis 1 examination

Enculturation can be less successful than expected. I will pick up an examination question of the final examination of a first-semester Analysis course for mathematics majors and future Gymnasium teachers. A systematic analysis of students results in the complete examination will be published elsewhere.

The task from the examination is shown in Figure 1.

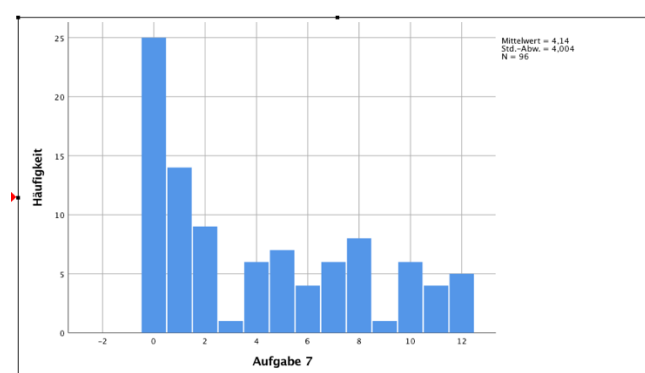
*Aufgabe 7 (my translation)*

Consider the Function  $F: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \int_1^x t^3 \exp(t) dt$ .

- (a) Prove that  $F$  is differentiable and  $F'(x) = x^3 \exp(x)$ . (4 points)
- (b) Determine all local minima and all local maxima of  $F$ . Determine the most extended possible intervals in which  $F$  is monotonically increasing and the most extended possible intervals in which it is monotonically decreasing. (8 points)

**Figure 1: The examination task**

The task is very close to school mathematical tasks but also has essential differences. The results were surprisingly bad from the perspective of the lecturer and the researchers. The average number of points was 4.14 of a maximum of 12 possible points (see Figure 2). The written test had nine tasks; five of them had better average results than task 7.



**Figure 2: Distribution of points of the examination task (N = 96), mean no of points: 4.14 of 12)**



$$\begin{aligned}
 a) \int_1^x t^3 \exp(t) dt &= t^3 \exp(t) \Big|_1^x - \int_1^x 3t^2 \exp(t) dt \\
 &= t^3 \exp(t) \Big|_1^x - 3 \int_1^x t^2 \exp(t) dt + \int_1^x 6t \exp(t) dt \\
 &= t^3 \exp(t) \Big|_1^x - 3 t^2 \exp(t) \Big|_1^x + 6t \exp(t) \Big|_1^x - 6 \int_1^x \exp(t) dt \\
 &= \exp(t) \Big|_1^x \\
 &\vdots
 \end{aligned}$$

### Aufgabe 7)

Betrachte die Fkt.  $F: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto \int_1^x t^3 \exp(t) dt$

(a) Zeige, dass  $F$  differenzierbar ist mit  $F'(x) = x^3 \exp(x)$ .

$$\begin{aligned}
 \int_1^x t^3 \exp(t) dt &= t^3 \exp(t) \Big|_1^x - \int_1^x 3t^2 \exp(t) dt \\
 \int_1^x 3t^2 \exp(t) dt &= 3t^2 \exp(t) \Big|_1^x - \int_1^x 6t \exp(t) dt \\
 \int_1^x 6t \exp(t) dt &= 6t \exp(t) \Big|_1^x - \int_1^x 6 \exp(t) dt = 6t \exp(t) \Big|_1^x - 6 \exp(t) \Big|_1^x \\
 \Rightarrow \int_1^x t^3 \exp(t) dt &= t^3 \exp(t) \Big|_1^x - (3t^2 \exp(t) \Big|_1^x - (6t \exp(t) \Big|_1^x - 6 \exp(t) \Big|_1^x)) \\
 &= [x^3 \exp(x) - \exp(1)] - ([3x^2 \exp(x) - 3 \exp(1)] - [6x \exp(x) - 6 \exp(1)]) \\
 &\quad - [6 \exp(x) - 6 \exp(1)] = [x^3 \exp(x) - \exp(1)] - [3x^2 \exp(x) - 3 \exp(1)] \\
 &\quad - [6x \exp(x) - 6 \exp(x)] \\
 &= x^3 \exp(x) - \exp(1) - (3x^2 \exp(x) - 3 \exp(1) - 6x \exp(x) - 6 \exp(x)) \\
 &= x^3 \exp(x) - 3x^2 \exp(x) + 6x \exp(x) + 6 \exp(x) + 2 \exp(1) \checkmark \\
 (x^3 \exp(x) - 3x^2 \exp(x) + 6x \exp(x) + 6 \exp(x) + 2 \exp(1))' &= \\
 = (x^3 \exp(x))' - (3x^2 \exp(x))' + (6x \exp(x))' + (6 \exp(x))' + 0 \\
 = (3x^2 \exp(x) + x^3 \exp(x)) - (3x^2 \exp(x) + 6x \exp(x)) + (6x \exp(x) + 6 \exp(x)) \\
 + 6 \exp(x) &= 3x^2 \exp(x) + x^3 \exp(x) - 3x^2 \exp(x) - 6x \exp(x) \\
 + 6x \exp(x) + 6 \exp(x) + 6 \exp(x) &= \underline{x^3 \exp(x)} \checkmark \\
 \Rightarrow F'(x) &= \underline{x^3 \exp(x)} \quad \text{Simplifiziert} \quad 4/4
 \end{aligned}$$

**Figure 3: Two example solution for calculating the integral and then the derivative, one of them with success.**

Subtask a) is a trivial one if one has understood the theoretical claim of the fundamental theorem of calculus. However, most students who worked on the task tried to calculate the integral (often not successful, see Figure 3 for the first example), and then they tried to calculate the derivative. The task a) would not have been posed in school mathematics, where the notion of differentiability is downplayed if at all mentioned and the theoretical nature of the FTC is downplayed in favour of its practical value to calculate integrals. One interpretation of this comparably unfortunate result is that students did view the FTC still from a school mathematical perspective and interpreted the task just as a calculation task.

Subtask b) is largely a school mathematics task, which, however, has to be solved on the level of university mathematics. Calculations show that  $F'(x) = 0$  iff  $x = 0$ , but  $F''(0) = 0$ . In order to argue

that there is a local minimum, students could reason with the sign change of  $F'(x)$  in  $x = 0$ , or calculate further derivatives finding that the first derivative with  $F^{(n)}(0) \neq 0$  is with  $n = 4$ , where  $F^{(4)}(0) > 0$ . This result can be used as an argument that there is a minimum of  $F$  in  $x = 0$ . However, these conditions based on higher derivatives are often not part of the university course. Our students had difficulties in various respects. For instance,  $F''(0) = 0$  is wrongly interpreted in that there is no extremum in  $x = 0$  (that this is a wrong conclusion is discussed in most school textbooks already). Another source of problems is that the statement " $F'(x) = 0$  iff  $x = 0$ " is often not interpreted as a theoretical result stating that can be no other extrema beside  $x = 0$ . In current school teaching, the condition  $F'(x) = 0$  mainly has the practical function to enable the calculation of extremal points. This tradition is challenged – but has probably not yet fundamentally changed -- by the widespread introduction of graphical calculators. If one "sees" a minimum in a function graph in a certain window and this seeing is allowed as an argument in students reasoning, it becomes more important to know that further extremal points cannot exist theoretically. So, graphical calculators have the potential to show the need for theoretical argumentation. However, these tools are not introduced in schools for this purpose.

Anyway, let us focus on the university context again. Explicit use of the theoretical nature of " $F'(x) = 0$  iff  $x = 0$ " in the argumentation, however, is necessary to give a complete answer to the question to find **all** extremal points.

We can only speculate what changes in the teaching of the Analysis course may lead to better results

- (a) The distinction of the theoretical and the practical value (for calculations) of a theorem has to be explicitly introduced.
- (b) In cases where university mathematics overlaps with school mathematics (such as in the case of finding extremal points), it may pay to explicitly contrast and relate the school mathematical reasoning to the university mathematical reasoning. This would help students see school mathematics from a higher standpoint, concretized by examples and not just as lecturers' lip-service to globally justify why Analysis is different from Calculus. With such examples, theoretical justifications of school mathematical practices including specifications of the application conditions of school mathematical practices (discussing extreme and other exceptional cases) can show the new university praxeology without neglecting the school level one.

Aspect (a) can be more easily included in an Analysis I course than (b). Whereas (a) makes explicit some implicit socio-mathematical norms of a course. The inclusion of considerations of (b) would imply that the comparison of school mathematical thinking and reasoning and university mathematical reasoning becomes a topic of a university mathematics course itself.

In school teaching, the explicit discussion of errors, preconceptions and misconceptions of students is seen as relevant, when conceptual change is the aim. This can be considered as a challenge of university mathematics, too. At least, the above example shows that school mathematical practices seem to coexist in students' minds together with new university mathematical practices and can lead to poor results in examination.

Discussing these changes from school to university mathematics has become the topic of some (non-obligatory) pre-university bridging courses. It is also a topic in course for future teachers on the didactics of calculus at the school level, in order to support the second transition back to school but does this does not help with the first transition and discontinuity. However, I know of no Analysis 1 course that has taken up this challenge.

### **Supporting students in developing adequate concept definitions and concept images at university: The case of the convergence of sequences**

In her dissertation which I supervised, Laura Ostsieker (2018) developed a non-compulsory 4-hour workshop for students of an Analysis 1 course that should support them in constructing the formal definition of convergence of a sequence in an attempt of guided reinvention. The workshop was situated some weeks after the start of the Analysis 1 course, just before the formal definition was introduced in the lecture. The study builds on work by Roh (Roh, 2010a, 2010b; Roh & Lee, 2017) and Oehrtman (Oehrtman, Swinyard, & Martin, 2014; Oehrtman, Swinyard, Martin, Roh, & Hart-Weber, 2011). The basic idea – which is taken from Przenioslo (2005) -- for setting up a learning environment is to give students a set of examples of sequences that are called to be “convergent” and others that are called non-convergent and ask the students to construct a definition of “convergence” where the examples are examples of, respectively non-examples (see initial task formulation in Figure 4). The set of examples was chosen to include examples that often have not become part of students’ concept image after students have learned the formal definition. Examples include sequences that are alternating around their limit or where some (a finite or infinite number of) elements are equal to the limit itself. Figure 5 shows the set of examples and one non-example. Scatter plot graphs of the sequences  $(n, a_n), \dots (n, x_n)$  were presented to the students, and the students were asked to relate the formal definition to the respective graph and to “formulate characteristic properties of every sequence”. After that the task was

*“The sequences  $(a_n) \dots (f_n)$  are called convergent to the limit 1; the sequence  $(x_n)$  is not convergent. Describe the joint property as good as possible that the sequences  $(a_n) \dots (f_n)$  have and the sequence  $(x_n)$  has not.”*

**Figure 4: Initial task formulation (Ostsieker 2018, p. 86; my translation)**

Laura Ostsieker had prepared a set of prompts for supporting the students when or if they got stuck. The workshop was offered twice. The second time, the revised conception of the workshop (tasks, visualisations, prompts) was offered to a different group of students one year later. The research study followed a design-based research framework. Hypothetical learning trajectories were developed for the first workshop, revised for the second one. Results from the retrospective analysis of the second version were provided for future implementation together with developing a local instruction theory. We cannot present Ostsieker with her detailed research questions and great results in this paper, but we will focus on one aspect: How did the formulation of the task change from the first to the third version? How does this change reflect some of the results of Laura Ostsieker’s empirical study?

$$(a_n)_{n \in \mathbb{N}} = \left( \frac{n+1}{n} \right)_{n \in \mathbb{N}}$$

$$(b_n)_{n \in \mathbb{N}} \text{ mit } b_n = \begin{cases} 1 - \frac{1}{n} & n \leq 125 \\ 1 & n > 125 \end{cases}$$

$$(c_n)_{n \in \mathbb{N}} \text{ mit } c_n = \begin{cases} -3 & 200000 \leq n \leq 500000 \\ 1 & \text{sonst} \end{cases}$$

$$(d_n)_{n \in \mathbb{N}} \text{ mit } d_n = \begin{cases} 1 & n \text{ Vielfaches von } 10 \\ 1 + \frac{1}{n} & \text{sonst} \end{cases}$$

$$(e_n)_{n \in \mathbb{N}} \text{ mit } e_n = 0, \underbrace{9 \dots 9}_n \quad \forall n \in \mathbb{N}$$

$$(f_n)_{n \in \mathbb{N}} \text{ mit } f_n = \begin{cases} 2 & n = 10 \\ 1 + \left(-\frac{1}{2}\right)^n & \text{sonst} \end{cases}$$

$$(x_n)_{n \in \mathbb{N}} \text{ mit } x_n = \begin{cases} 2 & n \text{ Vielfaches von } 10 \\ 1 + \frac{1}{n} & \text{sonst} \end{cases}$$

**Figure 5: The set of sequences  $(a_n) \dots (f_n)$  are examples for the concept to be defined,  $(x_n)$  is a non-example; “Vielfaches” means multiples; “sonst” means “else” (Ostsieker 2018, pp. 84).**

The task was challenging on two levels. We anticipated problems within the transition from an intuitive notion and verbally formulated attempt to characterize “joint properties” to a formal definition for the concept of convergence – given the very long history in mathematics that finally led to a formulation of the modern formal definition. What we did not anticipate were students' problems on a meta-level (see also Schüler-Meyer, 2018 for the role of the meta-level). They had to decide when a characterization they had developed with verbal and symbolic elements should be accepted as satisfactory. We had prepared the prompt “check whether your characterization is clear enough that it can be decided for every sequence whether it fulfils the characterization or not” to motivate and direct students for doing the next steps. In the second version of the task, we added this to the task itself to support a more self-regulated process of concept definition.

In retrospect, this was not surprising. Although the students had observed for several weeks how a mathematician was defining concepts and was using these definitions in theorem proving, students themselves had had no experience at all with activities of “defining concepts”, which is typical when mathematics is presented like a ready-made product and not as a process.

Results of Ostsieker's empirical study include a new formulation of the initial task and further elaboration of the support measures for the different steps of the process. I will cite the final third formulation and use it as a starting point for describing some of her results. This third formulation is a result of the retrospective analysis, but it was not yet tested empirically in a new trial.

1. *"We are going to discover the definition (in university mathematics) of the convergence of a sequence to 1. The concept has to be defined in such a way that the sequences  $(a_n) \dots (g_n)$  are convergent, and the sequence  $(x_n)$  is not.*
2. *Aim at formulating a joint property of the sequences  $(a_n) \dots (g_n)$  that the sequence  $(x_n)$  does not possess as one single condition.*
3. *This condition has to be formulated in such a way that somebody to whom this formulation is presented can objectively decide and argue for every arbitrary sequence, whether this sequence has the property or not. Every person to whom this formulation is presented should come to the same conclusion about every specific sequence.*
4. *If you have the opinion that you have formulated an adequate property, check for all the example sequences whether they possess this property and check whether  $(x_n)$  does not have this property. If some of these checks are negative, revise your formulation."* (Ostsieker 2018, p. 542, my translation. The numbering was added to make references in the following text clearer).

**Figure 6: final task formulation after two iterations of the workshop**

The formulation in 1. explicitly asks for a mathematical definition and not only for a "property" so that it is possible to refer to socio-mathematical norms for definitions when students present formulations of a property that does not qualify for a definition. A "definition in university mathematics" was added because some students had entered the debate with a school-mathematical concept of "convergence" according to which some examples were not convergent that were claimed to be "convergent" by the task. This happened when students had a school-mathematical convergence concept that calls sequences convergent only when the elements are approaching a limit (monotonically) but never reaching it. We did not expect that some students did not accept the rules of our "game", so we choose to make it explicit that we have to distinguish by name possible previous concepts from the new concept that is to be defined. This reformulation may help; however, self-confident students maybe not willing to give up a concept that they found useful in the past. This observation may point to limitations of the approach Laura Ostsieker (2018) and Przenioslo (2005) have chosen. The approach does not provide problems that motivate a change of previous definitions because of theoretical or practical reasons but forces the students to accept that there is some hidden good reason in the choice of the provided examples. In the sense of Freudenthal's didactical phenomenology, no actual problem situation is presented that motivates the creation of a new concept (Freudenthal, 2002).

The formulation in 2 ("one single condition") reflects situations that occurred in the students' debates. Some groups characterized several subgroups of the examples adequately and then formulated the definition by several statements connected by "or". For instance, "convergence" can mean

monotonically approaching a number with arbitrary closeness but not reaching it" or "being equal to this number from a certain  $n$  onwards" or "alternating around that number with ..." or doing one of these things with a finite number of exceptions" etc. This solution even if it fulfils all criteria of rigour, would be considered as "inelegant" in mathematics. This phenomenon reveals another norm that mathematical definitions have to fulfil. Students in the first two workshop were right in claiming that this requirement was not specified in advance. That is why it was added.

The formulation 3 reacts to the situation that some students were satisfied with formulations for themselves. Pointing to potential "readers" of the definition should initiate a reflection of the preciseness and understandability of the definition and motivate students to check their definition with further self-created examples or use other students' views. This extension of the task formulation may be of practical value. However, students may need to have more opportunities to reflect themselves on criteria that a mathematical definition must fulfil (Edwards & Ward, 2004; Ouvrier-Buffet, 2006; Zaslavsky & Shir, 2005). The formulation bears another problem. A mathematician would say that a mathematical definition must "in principle" allow to decide whether an object fulfils it or not. It will of course often happen that to decide whether a concrete object fulfils a definition or not may be considered as a severe mathematical problem to be solved in the future. Moreover, the formulation "every person" should come to the same conclusion is of limited practical: how can this be decided?

The formulation 4 reflects the following observation. In the anticipated learning trajectories, it was assumed that the students work on a successive improvement of their formulations of a definition, creating a first version of the definition from a limited set of examples, then systematically testing this formulation on all initial examples and also on new ones, then revising the definition and so on. Such a systematic approach was seldom observed. Therefore, the formulation was added to foster such an approach. The reason for the relatively unsystematic approach towards concept definition may not only be due to limited experiences in creating mathematical concept definitions, but it is plausible that students may have seldom met concept definition tasks or situations that require new concept definitions in their school and everyday life. In sum, these results also support the need to make implicit norms explicit and provide more extensive experiences for students to actively and reflectively participate in the new culture.

## **Concluding remarks**

The transition from school Calculus to university Analysis is complex and multifaceted. Our examples pointed to problems that more explicit enculturation into mathematics at university level may help to overcome.

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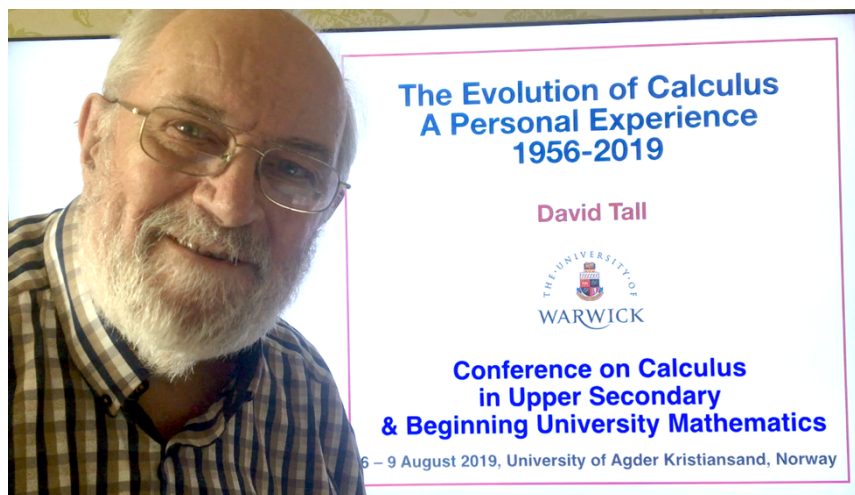
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## Plenary paper

### The Evolution of Calculus: A Personal Experience 1956–2019

David Tall

University of Warwick, UK; [david.tall@warwick.ac.uk](mailto:david.tall@warwick.ac.uk)



This presentation was prepared as a 30-minute opening plenary for the *Conference on Calculus in Upper Secondary and Beginning University Mathematics* at The University of Agder, Norway, August 6 – 9, 2019. I was invited to present a personal view of the development of calculus over the last fifty years to set the scene for the conference. For health reasons I was unable to travel, so I made the video at home. It is available on YouTube at

<https://www.youtube.com/watch?v=eOwQIEPKCfY&feature=youtu.be>

At the conference, participants will present various different approaches to the calculus. This opening plenary is designed to respect all viewpoints and to encourage participants to reflect on their personal views, to seek a broader overview of the whole enterprise. The issues are complex and no single individual can have a complete overall picture. So, although I will fulfil the request to present a personal view of the evolution of calculus, I will present evidence from a range of sources to seek an overall view. I will also add extra detail [in smaller type].

The calculus focuses on how we humans perceive, interpret and predict *change and growth*. The presentation will consider how we give meaning to dynamic change, how we interpret operational symbols as operations which then function as mental objects that can themselves be manipulated. It formulates how mathematics develops in sophistication through practical, theoretical and formal ways of thinking. This will include very recent developments in my own publications that offer simple new ways of giving meaning to symbolic expressions and new ways of interpreting dynamic change using retina displays.

I will begin with the human capacity for *embodiment* as we interact and interpret the world through practical activities and move to a higher theoretical level where visual embodiment of the limiting process stabilizes visually on the limit object. I will take an excursion to formal analysis where I will prove a simple formal theorem that shows that we can visualise infinitesimals as points on the number

line. I will use the cultural theory of Raymond Wilder to interpret how social aspects colour our ways of thinking and invite you to reflect honestly on your own viewpoint. I invite you to produce evidence to challenge and improve the overall picture that I offer. We make advances by challenging our own thinking.

## Introduction

This presentation reviews the rapid evolution of calculus over the last half century. It formulates simple ways to make calculus meaningful to the wide range of teachers, learners and experts that take account of the latest developments in digital technology, the workings of the human brain and the cultural aspects that affect the nature of human society.

The conference organisers invited me to offer a personal view of the evolution of calculus as it was 50 years ago and as it is now. I have decided to broaden this, to experiences since I first encountered calculus myself as a fifteen-year-old in school.

I note that the conference is focused around four themes.

- A. The school-university transition with a focus on calculus.
- B. The fundamental theorem
- C. The use of digital technology
- D. Teaching & learning in various disciplines.

This will be a personal journey. It will touch on ideas that have fascinated me as we live through a life-time that has seen – and continues to see – unprecedented evolution of digital technology. From such a vast enterprise, I seek an inner simplicity that gives human meaning to the mathematics of change and growth.

Let me begin by taking you back to my first encounter with calculus as a fifteen-year-old schoolboy. When I began to learn calculus, my mathematics master, Mr J. H. Butler, a graduate of Oxford University, wrote:

A handwritten note in blue ink on a light background. The text is written in a cursive, slightly slanted script. The first line reads 'If his effort matched his enthusiasm' and the second line reads 'he could be a competent mathematician'. The words 'enthusiasm' and 'mathematician' are highlighted in yellow.

Figure 1: First report as a calculus student

He was concerned that I spent too much time on a variety of activities and did not spend all my time on mathematics. This is still true and it is why I wish to refer to a range of other disciplines in my presentation, because we gain a greater insight into the ways we make sense of the calculus by considering the wider picture.

I really enjoyed working from the Calculus book by school teachers Durell and Robson, published in the thirties.

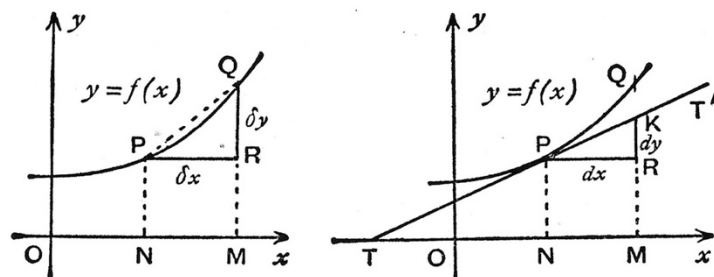


Fig 2: Introducing the derivative in the book by Durell and Robson (1933)

It defined the derivative as the limit of  $\delta y$  over  $\delta x$  as  $\delta x$  tends to zero. Much later, for any real number  $dx$ , it defined  $dy$  to be  $f'(x)$  times  $dx$ , then, for  $dx$  non-zero,  $f'(x)$  equals the quotient  $dy/dx$ .

This is based on the book by the Cambridge mathematician, Robert Woodhouse, published in 1803, which introduced the Leibniz notation to England after more than a century using mainly Newton's approach. It spread to English speaking nations around the world, including the American Mathematical Society which based its organisation on that of the London Mathematical Society in the 1880s.

Note the problematic meaning of the limit where  $\delta y$  over  $\delta x$  could only be calculated for  $\delta x$  non-zero while the limit value involved putting  $\delta x$  equal to zero.

There were also multiple meanings of symbolism, such as  $f'(x)$  being both a quotient and the symbol  $d/dx$  being an operation:

$$f'(x) = \frac{dy}{dx} = \left(\frac{d}{dx}\right)y$$

It was even an operation that could be repeated to give

$$\left(\frac{d}{dx}\right)^2 y = \frac{d^2 y}{dx^2}.$$

Then there was the meaning of  $dx$  in  $dy/dx$  and in the integral  $\int y \, dx$ , where it is spoken as *the integral of y 'with respect to x'*.

Many problematic meanings still remain.

### Problematic meanings in calculus today

Calculus is still based on an informal limit concept. For instance, the US Advanced Placement Calculus speaks of four 'big ideas':

- **Limits**
- **Derivatives**
- **Integrals and the Fundamental Theorem**

with an alternative curriculum including

- **Series**

The detailed curriculum focuses on concepts that can be *tested* on multi-choice tests.

Meanwhile, mathematics education research, is full of misconceptions that occur in learning.

Digital technology offers new facilities.

- for performing fast and accurate numerical calculations

- for the manipulation of symbols
- for drawing dynamic pictures that offer visual and conceptual *meaning*.

## Graphic Calculus

My own work in the 1980s involved developing software called *Graphic Calculus* to visualise concepts.

One program allowed the user to magnify part of the graph, to see that, close up, familiar graphs looked less curved and — under high magnification — they looked *locally straight*.

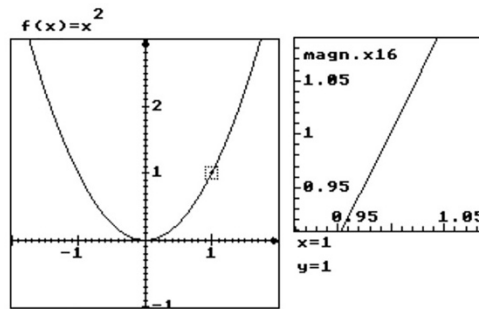


Figure 3: Magnify

*The derivative now is not introduced as a limit, it is the slope of the graph itself.*

It is possible to look along the graph to *see* its changing slope.

It is also possible to trace it with your hand to *feel* it.

It has human *meaning*.

Another program plots successive lines of slope  $(f(x+c) - f(x)) / c$  on the graph, for variable  $x$  and small fixed  $c$ , to plot points that lie on the graph that I termed *the practical slope function* for variable  $x$ . (In the UK we called it the ‘gradient function’.)

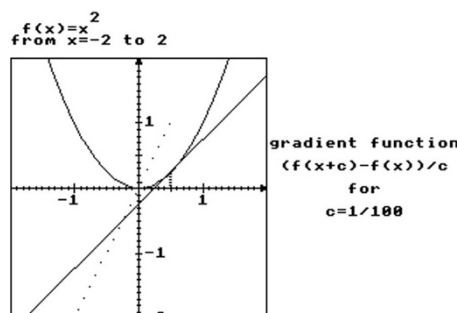


Figure 4: Plotting points on the practical slope function

In the case of  $f(x) = x^2$ , the practical slope function is  $2x+c$ . As  $c$  is taken to be small, the graph stabilises visibly to the graph of  $2x$ .

Local straightness gives meaning to all standard derivatives.

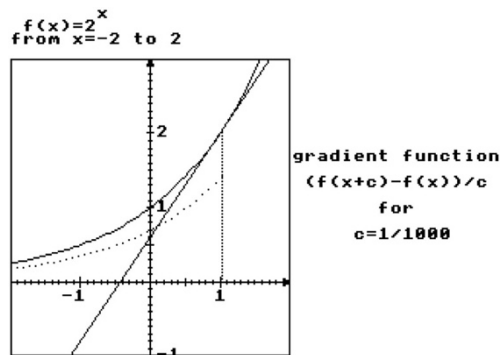


Figure 5: The practical slope function of  $2^x$

For instance, looking at the graphs of  $2^x$  and  $3^x$  shows that they have the same general increasing shape but the slope of  $2^x$  is below the original and  $3^x$  is above. Look for a number  $e$  between 2 and 3 where the slope function for  $e^x$  is again  $e^x$ .

Seeking a polynomial approximation links the visual information to symbolic calculation to give a *sense* of how quickly the approximation converges and allows  $e$  to be calculated to ten or more decimal places by hand. (Or you may use *Excel*, if you really wish to do so ...)

The graphic approach gives visual meaning to the symbolic derivative. 😊 (Wow!)

But it does not deal with the rules of differentiation. 😞 (Uhu!)

Recent developments which I will discuss later offer new meanings to fill this gap. This will offer a simple way to understand how spoken articulation gives meaning to expressions and how to interpret the flexible meaning of sub-expressions as process or mental object.

## Integration and the fundamental theorem

Now we turn to integration and the fundamental theorem. In this case, I take a function such as  $\sin(x)$  and keep the same vertical scale while stretching the horizontal scale. In this case you can see that the graph ‘pulls flat’ (Figure 6).

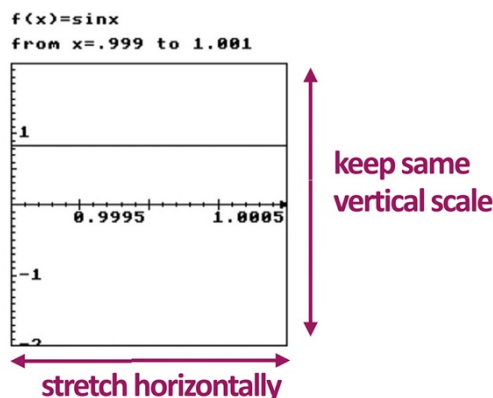


Figure 6: Stretch a (continuous) graph horizontally

Taking the vertical thickness of the pixel line as  $\pm\epsilon$ , the idea of ‘pulling flat’ embodies the formal definition of continuity at  $x$ :

Given any  $\epsilon > 0$ , we can find a  $\delta > 0$  such that, stretched in a window width  $x \pm \delta$ , the graph lies in the pixel line  $f(x) \pm \epsilon$ .

Drawing a continuous graph on paper involves a dynamic physical process, that produces a visual object: the graph. Suitably programmed, a digital picture of a continuous graph will ‘pull flat’ when it is stretched horizontally.

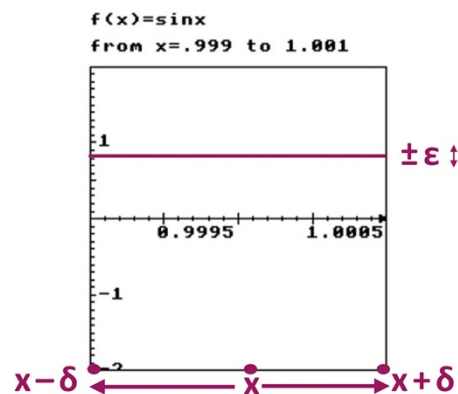


Figure 7: Dynamic definition of continuity

This offers a vital change in meaning. While a static picture in a book does not offer any insight into the fundamental relationship between practical continuity in drawing and the formal epsilon-delta definition, horizontal stretching of a digital picture of the graph offers a dynamic embodied meaning that links practical drawing of a continuous curve to the formal definition of continuity.

This offers a new embodied insight into the fundamental theorem of calculus (Figure 8).

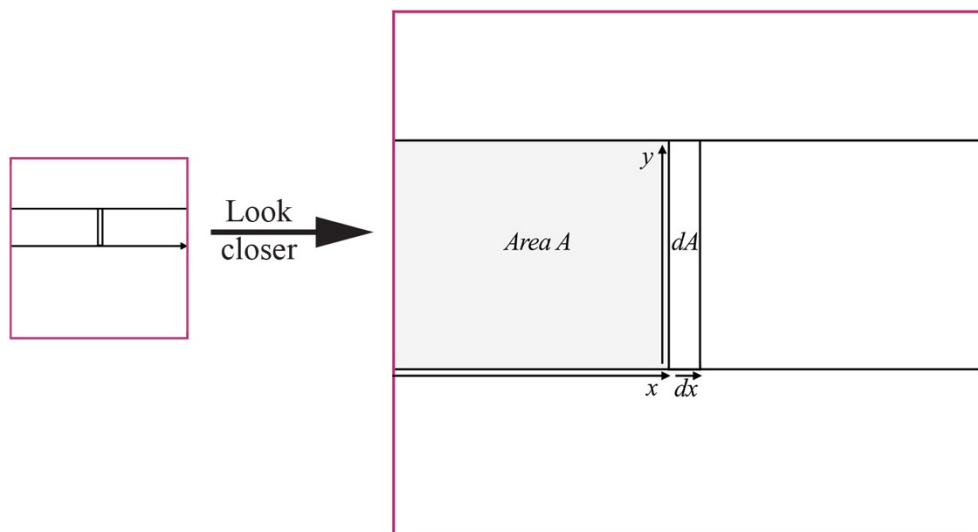


Figure 8: Visualising  $dA$  as  $y dx$

If we look closer at the picture where the graph has been pulled flat in a fixed window, the area  $A$  to a point  $x$  on the  $x$ -axis with  $y$  measured up from the  $x$ -axis can be imagined to change by an increase  $dx$  in  $x$ . Then the increase  $dA$  in  $A$ , calculated as  $y$  times  $dx$ , is given as  $dA = y dx$ .

This has the useful idea that it is based on a link between the embodied action of drawing and the formal definition of continuity. 😊 (Wow!)

But on the negative side there are concerns about the meaning. 😞 (Uhu!)

The horizontal stretching clearly changes the visual area of the strip width  $dx$ , height  $y$ . But if the product  $y dx$  is interpreted *as a number* then this remains unchanged. This clearly signals the need to realise that there are (at least) two distinct interpretations of the calculus. One involves the variables

as *numbers*, the other involves the variables as *quantities with dimensions*, such as time, length, area, volume, velocity, mass, temperature, density, pressure, national debt, share value, and any other measurable quantities that arise in applications.

In addition, there is still the question about the exact meaning and value of  $dA$  being precisely equal to  $y \, dx$ .

## Levels of Sophistication in Calculus and Analysis

To respond to this, we need to look at the bigger picture.

I suggest that **CALCULUS** has two successive levels of sophistication. One is the

**Practical** level that involves visual representation and symbolic calculation, in which a process *approaches* a limiting value.

Then there is the

**Theoretical** level that involves informal definition and deduction, focusing on the limit object itself.

**ANALYSIS**, on the other hand, is based on

**Formal** mathematics, using set-theoretic definition and formal proof.

Users of Calculus essentially only encounter practical or theoretical mathematics, which involve visual and symbolic aspects. Practical mathematics involves coherent ideas that fit together. Theoretical mathematics introducing more coherent forms of reason, where specific assumptions have necessary consequences.

Human perception and action involve more than visual input. More broadly, they involve a wide range of perceptual senses and forms of action and reason that I term **Embodiment**. This is physical and mental, visual and gestural, and includes mental thought experiments.

**Symbolism** involves calculation, compressing processes in time as operational symbols that can be mentally manipulated as objects, where the symbols have multiple meanings.

The Practical limit is *a process that gets as close as is desired*.

The Theoretical limit is the limit *object*.

Embodiment suggests the limit is visually attained. 😊 (Wow!)

Symbolism suggests that the limit is not attained. 😞 (Uhu!)

## Cultural aspects of mathematics

Mathematics arises in different cultural settings. I build my framework on the insight of mathematician Raymond Wilder who used the anthropological term ‘culture’ to study the evolution of mathematical concepts.

He wrote:

[A culture is] a collection of customs, rituals, beliefs, tools, mores, and so on, called cultural elements, possessed by a group of people who are related by some associative factor (or factors) such as common membership in a primitive tribe, geographical contiguity, or common occupation.



He used a number of anthropological terms:

- **Cultural stress** involves a need in the community that requires to be resolved
- **Cultural diffusion** refers to moving cultural elements between cultures
- **Cultural lag** acknowledges that diffusion takes time
- **Cultural resistance** occurs when a community opposes diffusion.

To these I add a positive heading to balance the negative notion of resistance:

- **Cultural stability** involves maintaining a culture that seems to be working.

### *How do these cultural aspects affect calculus?*

In the calculus we have:

- **Stress from different needs in different communities**
- **Stress from fast changing digital technology**
  - **Cultural lag in transmitting new ideas to communities that may not even be aware of them**
- **Resistance to new ideas competing with stability**

*A major cultural factor is the conflict between cultural stability, and changing need, which underlies the so-called math wars.*

Pure mathematicians seek formal proof. Users require theoretical mathematics involving limit objects or practical mathematics involving limit processes. The plan is top-down, the learning is bottom up (Figure 9).

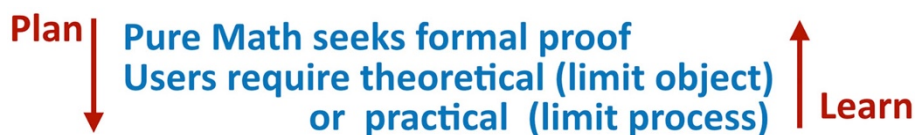


Figure 9: Directions of planning and learning

### **How does this relate to How Humans Think Mathematically?**

To gain insight into how we think mathematically, it is first useful to understand how we interpret what we see (Figure 10).

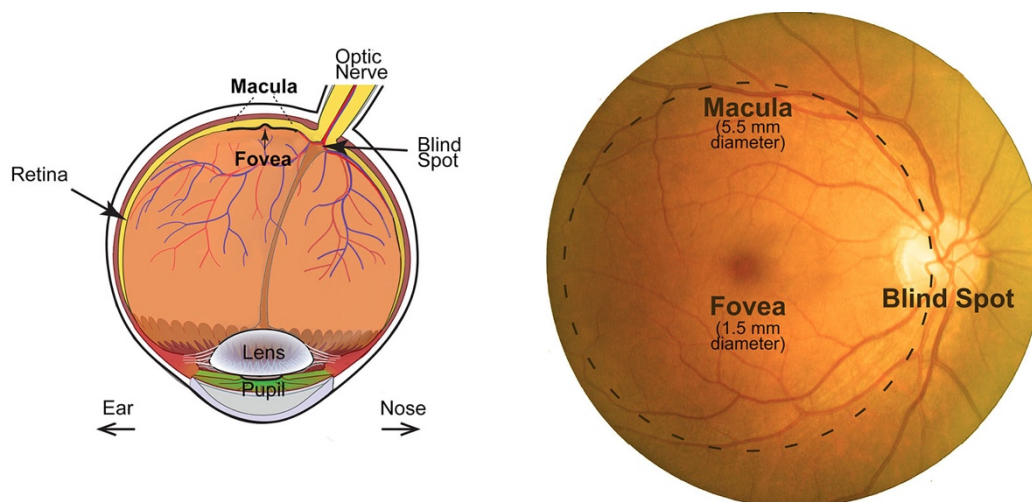


Figure 10: The structure of the eye

On the left is a cross section of the eye looking from above with the pupil at the bottom and the back of the eye at the top. On the right is a view of the central part of the back of the eye as seen through the pupil which includes an area called the **macula**, about 5.5 mm across. This recognises visual detail.

In the centre of the macula is circular region called the **fovea**. This recognises high resolution colour and detail through cells called **cones**. There are around 200,000 cones and the fovea is only about 250 cones across. **We will see that this has a profound effect on how we read symbols and how our eyes follow moving objects.**

When we read text, the eye stops for a brief *fixation* to take in information through the fovea and shifts between fixations in jumps called *saccades*. Read the words on this page to get a sense of how your eye jumps in saccades and stops momentarily in fixations.

Do this now, so that you have a *sense* of what I am talking about ...

## Reading mathematical expressions

Expressions are read in many languages left to right in chunks.

For instance, this expression

$$1 + 2 \times 3$$

is read as **one plus two times three**.

The meaning depends on the spoken articulation. If I leave small gaps between symbols, then the meaning can be made more precise. Here I use three dots (an ellipsis ...) to denote a brief gap in speech so that “1 + 2 ... times 3” means that I say “1 + 2” (which is “3”) then, after a gap (...) I say “times 3”. In this case,

“1 + 2 ... times 3” is **3 times 3**, which is **9**.

Meanwhile,

“1 + ... 2 times 3” is **1 plus 6**, which is **7**.

Once this meaning is understood, the distinction can be written symbolically using brackets as **(1+2) × 3** and **1 + (2 × 3)**.

This leads to:

**The Articulation Principle:** The meaning of a sequence of operations can be expressed by the manner in which the sequence is articulated.

This *Principle of Articulation* can be used at any level to give meaning to the use of brackets. It is not a mathematical definition, but it does lead to meaning for the use of symbolic operations throughout mathematics.

I have been developing this idea in recent years in invited papers that are available in draft form from my academic website: (<http://homepages.warwick.ac.uk/staff/David.Tall/downloads.html>) where they are listed from Tall 2017a to Tall 2020a. For details interpreting operational symbolism, see, for example, [Tall, 2020a](#), pp. 10–13.

These represent a continual theoretical development with much in common, but each one is written afresh for different audiences. I am currently working on an overall paper devoted to calculus. When completed, it will be added to my academic website downloads page.

### How does the eye and brain perceive motion?

To follow a moving object, the mechanism of the eye again works in saccades and fixations. Reading text involves a succession of fixations to read information in chunks separated by quickly jumping saccades. Following a moving object happens in a different way. There is an initial saccade to fix on the object and then the fixation moves smoothly to follow the object.

Place your finger in front of you and move it to the left and right. The eye makes an initial saccade to focus on it, then moves in a smooth fixation to follow it. This will happen if you move your head or keep your head still and just turn your eye.

Try this out for yourself, moving your head or keeping it still, to see how you focus smoothly on your moving finger while the background is blurred.

Do this now to sense how it happens ...

Now imagine two points on a number line, one fixed, one moving (Figure 11).



Figure 11: Variables and constants

In the presentation, the moving point is animated to move towards the fixed point, to encourage the viewer to imagine a variable point moving towards a fixed point. This reveals the idea that the brain naturally imagines constants and variables, including quantities that can become arbitrarily small.

***The intuitive foundations for calculus are built into the human brain.***

### Zooming in on a curve on a retina screen

Looking back at the original picture from the 1933 text of Durell and Robson, it becomes evident that looking at this picture gives no sense of the local slope of the graph. Indeed, if one magnifies part of the picture, this simply thickens the lines. Figure 12 shows the original picture magnified in a box where the box remains the same size and the picture is magnified within it.

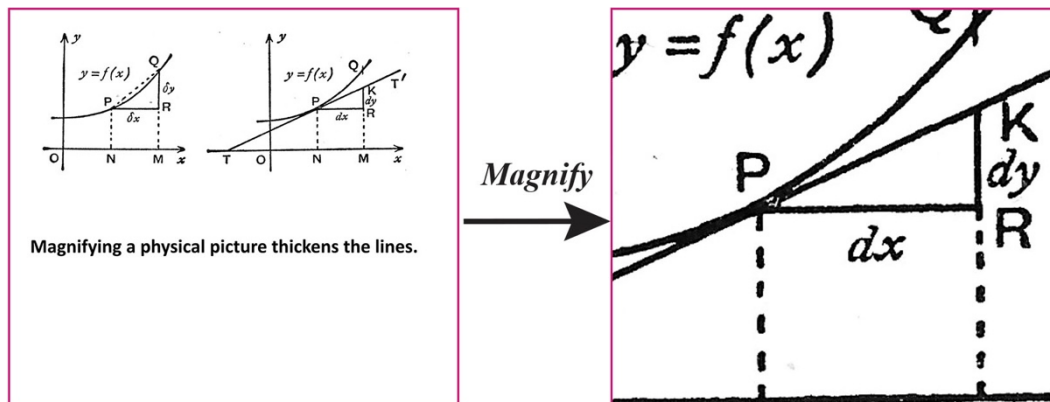


Figure 12: Magnifying a physical picture in a book

Scaling on a retina display can be programmed to maintain the thickness. (Figure 13).

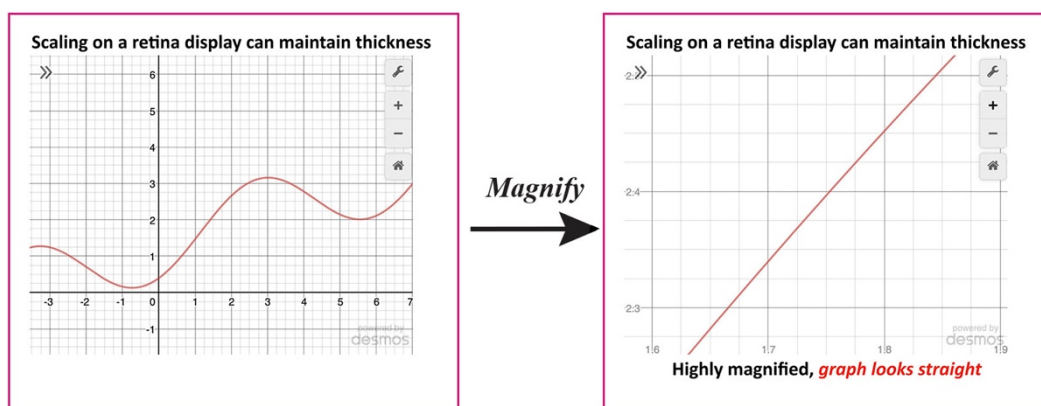


Figure 13: Magnifying a curve on a retina display

Highly magnified the graph looks like a straight line. By maintaining the thickness of the curve in the visual representation, the viewer can now see that the graph is locally straight. This changes the focus of attention from the *practical limit* which *approaches* the limit object to the *theoretical limit* which *is* the limit object.

***The notion of local straightness relates directly to the structure and operation of the human eye!***

Using a retina display, the practical slope function stabilises on the theoretical slope function.

This gives a visual embodiment supporting local straightness which switches attention from

**The practical limit** as a process that gets as close as is desired

to

**The theoretical limit** which is the derivative as a limit object.

*It gives meaning to the derivative as a theoretical object.*

So, we must ask the fundamental question:

***Why is local straightness rejected in AP calculus?***

## Cultural resistance over the centuries still operates today

A reason for the persistence of the traditional approach to the calculus (starting from an informal version of the modern definition) may lie in the desire for cultural stability and its consequent cultural lag and cultural resistance. New forms of technology provide radical changes communication, operation and imagination that cause great cultural stress by changing faster than current communities can cope.

This is a natural feature of the evolution of ideas as new generations build on the developments of earlier generations. Today we still have the vestiges of earlier beliefs that act as obstacles for change. For example, consider the following Greek common notion:

### **Greek Common Notion** **The whole is greater than the part**

This is self-evident for counting objects or measuring figures.

But Galileo was found guilty of heresy for suggesting that the whole numbers are in one to one correspondence with the squares of whole numbers.

Cantor realised that this property is the *definition* of an infinite set.

This was met with hostility and cultural rejection and led to his mental breakdown.

I pose the question:

*Is this is happening in Calculus today?*

## The meaning of infinitesimals

Let us consider the formal theorem:

### **Theorem** **A complete ordered field cannot contain infinitesimals**

This is self-evident for counting objects or measuring figures.

It implies there are no infinitesimals in the real numbers.

However, it does not imply that there are no infinitesimals on a number line ...

... just as there are *no irrationals* on the *rational number line*.

I can prove to you that infinitesimals must occur in *any* ordered extension field of the real numbers.

Let  $K$  be any ordered field with  $\mathbb{R}$  as an ordered subfield. Define an element  $x$  in  $K$  to be *finite* if it lies between two real numbers,  $a$  and  $b$  ...

... and to be *infinitesimal* if  $x$  is non-zero and lies between  $-t$  and  $t$  for every positive real number  $t$ .

Then we can prove the following structure theorem:

**Structure Theorem:** Any finite element  $x$  in  $K$  must either be in  $\mathbb{R}$ , or of the form  $c+\varepsilon$  where  $c$  is in  $\mathbb{R}$  and  $\varepsilon$  is an infinitesimal.

**The proof is a simple application of the completeness axiom.**

Call elements in  $K$ , ‘quantities’, and elements in  $\mathbb{R}$ , ‘constants’.

Then we have the following Theorem:

**Any finite quantity is either a constant  
or a constant plus an infinitesimal.**

For finite  $x$ , the unique real  $c$  is called *the standard part* of  $x$ , written  $\text{st}(x)$ .

To see infinitesimal detail, we magnify the scale by defining the  $\varepsilon$ -lens pointed at  $c$  for any  $c$  and non-zero  $\varepsilon$  to be

$$m: K \rightarrow K \text{ such that } m(x) = (x - c)/\varepsilon.$$

This magnifies the line by an infinite factor  $1/\varepsilon$  focused on the point  $c$ .

Define the field of view of  $m$  to be the set  $V$  such that  $(x - c)/\varepsilon$  is finite.

The optical  $\varepsilon$ -lens pointed at  $c$  is  $\mu: V \rightarrow \mathbb{R}$  given by

$$\mu(x) = \text{st}((x - c)/\varepsilon)$$

I introduced the notion of optical lenses in Tall (1982b). I proved the Structure Theorem for Infinitesimals in the form given here for an undergraduate course. I never published it as a mathematics paper because I considered it a simple exercise using the completeness axiom. When Ian Stewart and I wrote new editions of *Foundations* (1977) and *Complex Analysis* (1983) in 2014, 2018 respectively, we incorporated chapters on real and complex infinitesimals using this Structure Theorem. It is now an essential link between formal mathematics and more sophisticated forms of embodiment and symbolism.

This gives a map from the field of view  $V$  on the extended number line  $K$  to the real number line. When  $\varepsilon$  is an infinitesimal (in which case the  $\varepsilon$ -lens is also called an *optical microscope*) the original elements  $c, c + \varepsilon, c + 2\varepsilon, \dots$ , in  $K$  differ by infinitesimal quantities, but their images on the real line are visibly distinct. So, we can now ‘see’ infinitesimal differences under infinite magnification factor  $1/\varepsilon$  (Figure 14).

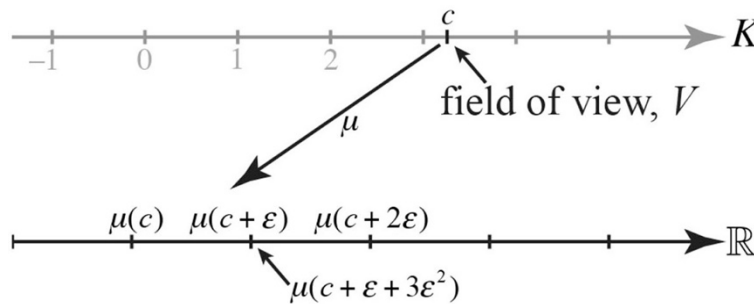


Figure 14: Optical  $\varepsilon$ -lens

However,  $c + \varepsilon$  and  $c + \varepsilon + 2\varepsilon^2$  differ by  $2\varepsilon^2$  and the optical lens maps them to the same point.

Further details are available in the second edition of *Foundations of Mathematics* (2014) written jointly with Ian Stewart and, in more detail for real and complex analysis in our joint *Complex Analysis* (2018).

Furthermore, the ideas apply in two or more dimensions, simply by using optical lenses on each component. An optical lens in  $n$  dimensions can use different magnification factors on each coordinate.

For example, the magnification factor  $1/\varepsilon$  can be used on both coordinates to see local straightness of a differentiable function (Figure 15).

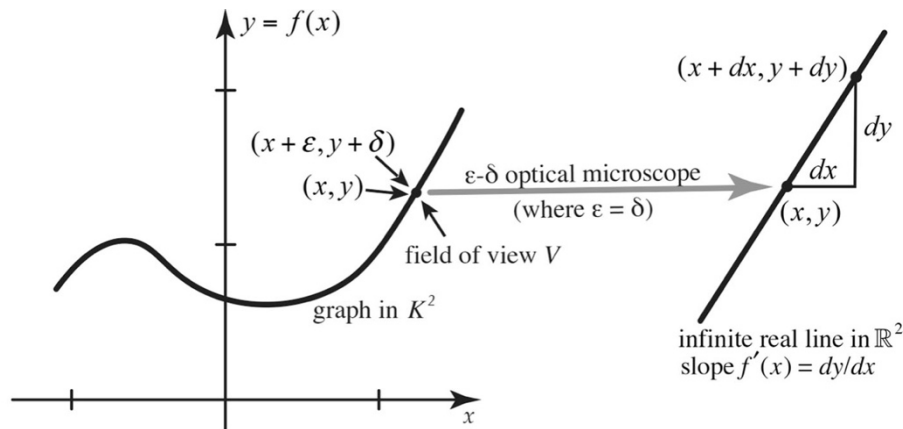


Figure 15: Local straightness of a differentiable function

A horizontal magnification of  $1/\varepsilon$  and a vertical magnification of 1 can be used to see local flatness of a continuous function (Figure 16).

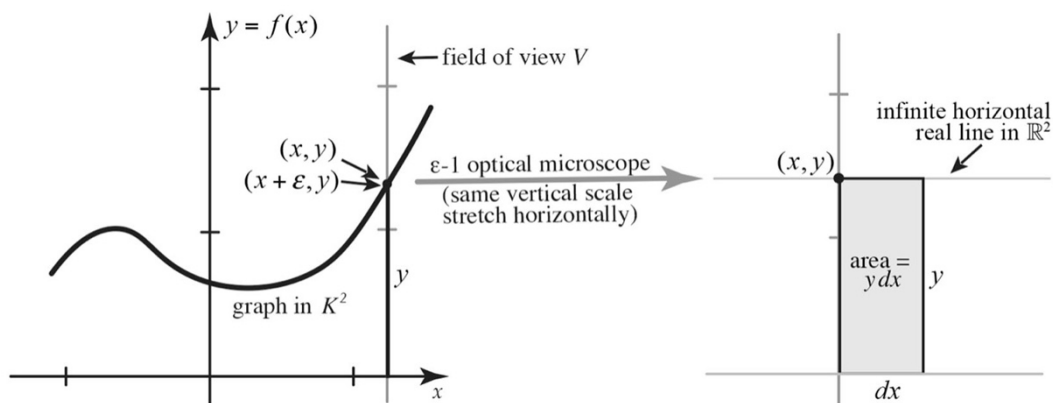


Figure 16: Local flatness of a continuous function

In both this pictures, I have taken the liberty to denote the image  $\mu(s, t)$  for  $(s, t)$  in  $V$  by the same symbol  $(s, t)$  in  $\mathbb{R}$ , even though the original coordinates  $s, t$  are in the field of view in  $K$  and the image coordinates are real numbers. This is a long-standing convention in making maps where we denote the image of, say, New York, on a map by the name ‘New York’. In the same way, when we draw a map, we only represent certain level of detail appropriate for the scale of the map.

The picture of local flatness (figure 16) appears here in this form for the first time. Other details can be found in chapter 13 of *How Humans Learn to Think Mathematically* (Tall, 2014).

Optical lenses arise in Formal Mathematics. They are included here to show formal mathematicians that infinitesimal ideas are a fully valid part of formal mathematics.

In my presentation, I did not have time to elaborate on important ideas presented in the second edition of *Complex Analysis*, chapter 15, where I put forward the argument that thinking of infinitesimals as ‘arbitrarily small variables’ is more productive than using formal optical lenses. The reason for this arises in dealing with polar coordinates where small increments  $dr$  in  $r$  and  $d\theta$  in  $\theta$  give a small approximate rectangle with side



lengths  $dr$  and  $rd\theta$ . Using optical lenses it is not possible to simultaneously magnify the infinitesimal shape with sides  $dr$  and  $rd\theta$  while also including the origin, when it is a finite non-zero distance  $r$  away. However, it is easily possible, *in the mind's eye*, to imagine a picture of the plane with the origin and the tiny shape in the same picture and then to zoom in on the shape to *see* it look like a rectangle. (This could also be represented appropriately on a retina display.) For this reason, applied mathematicians use an embodied human version of the calculus to make sense of arbitrarily small changes.

I also did not have time to respond to the claim that local straightness is inappropriate for calculus based on the idea that, when applied to polar coordinates, it no longer works. This arises from different views of the meaning of a graph in the plane, as to whether it is a two-dimensional object in  $\mathbb{R}^2$ , or a set of ordered pairs in  $\mathbb{R} \times \mathbb{R}$ . This confusion has lasted for centuries and it continues today in the distinction between  $y$  being a function of  $x$  in  $\mathbb{R} \times \mathbb{R}$  and the notion of covariation between variables  $x$  and  $y$  in  $\mathbb{R}^2$ . Both are valid, but with very different meanings. It has taken me many years to understand this, although there are vestiges of the distinction in my published writing. Back in 1977, in the first edition of *Foundations*, in a section written by Ian Stewart, there is an explanation of the notion of a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  which begins with two copies of the number line  $\mathbb{R}$  and then turns the second number line round through  $90^\circ$  with the arrow going from a point on the horizontal  $x$ -axis to the vertical  $y$ -axis being redrawn to go vertically from the point  $x$  to the height  $y$  and turning through a right angle to end up on the vertical axis at a height  $y$ . The relationship between  $x$  and  $y$  is then represented by the graph of points of the form  $(x, y)$  as  $(x, f(x))$ . Local straightness applies to this interpretation.

The notion of co-variation involves  $x$  and  $y$  varying together, with a relationship between the two. This can be considered as a change occurring *in time*. In this interpretation there is a hidden parameter  $t$ . It doesn't matter how fast  $t$  changes, *as long as it goes in the same direction*. This is related to Newton's idea of *fluxions* where variables change in time. Leibniz, on the other hand, essentially saw the curve as an object that was made up of an infinite number of infinitesimally short sides.

As individuals with different genetic inheritance and different previous experiences, we all develop different knowledge structures. Mine, as a Grammar School boy and an Oxford undergraduate and graduate, caused me to see the graph as a relationship relating the independent variable  $x$  to the dependent variable  $y$ . I saw the relationship as one-way and could not grasp the idea of balanced *co*-variation, because I believed that  $y$  varies as  $x$  varies, but not the other way round.

In dealing with implicit functions I was faced with a dilemma. I found the loose way that English text books dealt with a relationship such as  $x^2 + y^2 = 1$  to 'differentiate with respect to  $x$  to get  $2x + dy/dx = 0$ , and so  $dy/dx = -2x$  was, to me, unsatisfactory, because  $y$  is not a function of  $x$  in the previously defined sense. So, in designing the calculus curriculum for the School Mathematics Project, I began by introducing a parametrisation, in this case  $x(t) = \cos(t)$ ,  $y(t) = \sin(t)$  for  $0 \leq t \leq 2\pi$ , to calculate  $dx/dt$ ,  $dy/dt$ , where  $(dt, dx, dy)$  is the tangent vector in 3 dimensional  $(t, x, y)$  space. This is an alternative way of conceiving  $x$  and  $y$  as being co-variant in ordered time  $t$ , so that co-variation now makes sense to me. Local straightness still applies, but now it is in three dimensions. In fact, it naturally generalises to the multi-dimensional case, but that is another story.

Strangely, I now see Einstein, entering the picture that was imagined by Newton. Einstein lives in four-dimensional spacetime with three-dimensional space and a fourth time dimension that is *directional* and only operates in that one direction. In such a world view, co-variation is a natural concept:  $x$  and  $y$  co-vary in a particular direction *in time*. In our imagination we can see time as a video, say of a train travelling in reverse, then we play the video backwards to see the train travelling in a forward direction. Our imagination goes beyond our real-world experience.

Different viewpoints are enormously valuable. They enable us to see possibilities beyond our own personal experience. I have in recent years had the scales removed from my eyes to see such possibilities, but I still sense we need to stand *above* the differences to question our own views and to seek profound underlying patterns.

We can learn by reflecting deeply on serious data produced by various studies in a range of different areas of expertise. For example, the report on *Insights and Recommendations from the MAA National Study of College Calculus* by Bressoud, Mesa and Rasmussen (2015) can be downloaded from the internet at

<https://www.maa.org/sites/default/files/pdf/cspcc/InsightsandRecommendations.pdf>

and contains a wealth of detailed information.



Analysing the data from viewpoints in this presentation – including personal brain structure and operation, corporate cultural aspects and the long-term development of mathematical thinking – gives new ways of forming a coherent overview that reflects the realities of how we think mathematically, and how our cultural evolution affects our teaching, learning and thinking about calculus.

The question is: given the cultural pressures to maintain cultural stability in the face of cultural resistance, how is it possible to move ahead? My proposed solution in recent papers is, first to introduce principles such as the Principle of Articulation to give meaning to symbolism while linking embodiment and symbolism. Then I have analysed the changes in context throughout the curriculum to identify those aspects that remain supportive over several changes. These can give a longer-term sense of stability allowing the learner to reflect on those changes that are problematic so that they can be addressed meaningfully at the time that they arise.

But will it happen? This is not in my gift.

## Reflections

I have included the notion of optical lenses to show that they justify the notion of infinitesimals on a number line using formal mathematical theory. However, this theory belongs in formal mathematics using the axiom of completeness, not in the practical or theoretical levels of sophistication of calculus.

The idea of optical magnification can be visualized in Practical Embodiment in any program that allows embodied scaling on each coordinate. This is possible in GeoGebra.

The original TI-92 graphic calculator allowed zooming in with differing scale factors, based on my advice to the company, but this is not used in any curriculum as far as I know. The current implementation in GeoGebra allows magnification of the plane by moving finger and thumb apart and has the possibility of stretching horizontally by stretching the horizontal axis. In the version I have seen, it needs reprogramming to allow horizontal stretching when the horizontal axis is not in the picture.

I understand that currently there is competition between GeoGebra and the software Desmos for the larger market share. The GeoGebra interface operates on the picture itself to stretch it in various ways while Desmos prefers to use sliders to alter parameters. Desmos has the advantage of a broader classroom environment in which all the students work on their own smart phone, iPad or computer, connected using wi-fi. It is then possible to display the screen of any student on the teacher's main display to encourage class discussion. Desmos is therefore smarter in providing an environment for formative assessment of how different students are progressing as compared to the summative multi-choice tests currently used in AP Calculus.

Stretching graphs on a retina display supports the notion of infinitesimals as variable quantities as envisaged in mathematical applications and also as visualised naturally by the human eye and brain. Infinitesimals can be imagined as processes in practical mathematics and embodied as objects in theoretical mathematics. They make sense as 'arbitrarily small quantities' in applied mathematics.

In my presentation I focused on how we think about Calculus, in particular the roles of **Embodiment** and **Symbolism**. Embodiment includes how our brain makes sense of dynamic change, speaking and reading text and mathematical expressions, translating these human activities into operational symbolism. This symbolism involves *embodying* change through operations symbolised as mental objects that can themselves be operated on at a more sophisticated level.

I have identified three significant levels of sophistication:

- Practical limit processes
- Theoretical limit objects
- Formal limit definitions and proof

In the learning and teaching of these ideas, I propose that we build *up* to the limit concept through practical and theoretical experiences, not *down* from a formal definition, as yet unknown to the learner, presented in a manner that focuses on a process that seems unfinished.

In the limited time available, I did not include essential parts of my broader framework which incorporates affective aspects of mental activity, including confidence, anxiety and so on that are included in chapter 5 of *How Humans Learn to Think Mathematically* (Tall, 2013).

There I built on earlier work of Skemp, based on Freudian psychology where he formulated a theory of *goals* (to be desired) and *anti-goals* (to be avoided).

A goal that one believes is achievable is accompanied by a feeling of confidence, which may change to frustration if it proves subsequently to be difficult to achieve. Frustration sensed by a confident person is likely to act as a positive encouragement to redouble the effort to achieve the goal. Moving towards a goal gives pleasure and moving away from it gives unpleasure—a term used in Freudian analysis to denote the opposite of pleasure.

Coping with an anti-goal is quite different. According to Skemp, an anti-goal that one believes one can avoid gives a sense of security but, when it cannot be avoided, the emotion turns to anxiety. Moving towards a goal instils a sense of fear, while moving away from it gives relief.

This theory is represented in figure 17 (which I drew in Tall, 2013, p.120) where arrows represent movement to or away from a goal or anti-goal and smiling or frowning icons represent the belief related to the ability to achieve a goal or avoid an anti-goal.

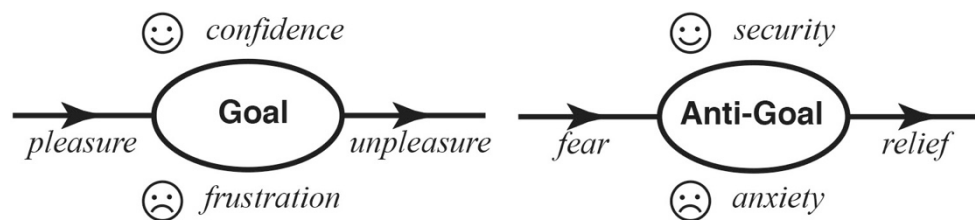


Figure 17: Emotional reactions to goals and anti-goals

This reveals the vast difference between positive emotions of confidence and pleasure relating to goals that are considered achievable and emotions relating to anti-goals which offer, at best, a sense of security and relief and, at worst, a sense of anxiety and fear.

Since 2013, I have studied the literature on neurophysiology more closely, comparing its studies of fMRI scans which only note changes over a period of two seconds or so, as compared with what can be observed in a classroom by an observer familiar with the structural operation of the brain. In particular, I have taken great interest in the operation of the limbic system which links to automatic body regulation and operates quickly before the frontal cortex can make a considered decision. Here the mathematician and neuroscientist Stanislaus Dehaene has provided some very interesting data recording the parts of the brain that respond to various forms of mathematical thinking and suggests that the brain recruits areas related to handling of number and shape and space to think mathematically that are not related to areas involved with language.

Without language there would be no sophisticated mathematics, for it is in *naming* ideas that we can begin to think about them and communicate them in increasingly sophisticated ways. However, as human beings our brains develop specialised functions that we pass on to later generations, accumulating mathematical knowledge. This relates to our natural abilities to repeat sequences of actions and to name operations using operational symbolism, leading to a compression of knowledge where the symbols become mental objects that can themselves be operated upon.

The question of meaningful and rote learning can now be considered in a different light. If rote-learning is prioritised over making meaningful links, then, in the long term, the compression of

knowledge is less likely to occur in ways that enable more sophisticated levels of mathematical thinking.

Cultural stability operates to preserve those aspects which are considered to be important. But does it serve us well in the long-term?

***I have proposed the need to think about cultural resistance  
that may impede our progress.***

This requires a reflection on the subtle aspects that underpin our personal thinking. I seek a deeper simplicity in my life. I have a few more years on this earth and I wish to spend it in quiet contemplation and have no desire to become involved in emotional disputes. I do welcome any comments that cause me to rethink my position as this can help me improve my understanding.

Meanwhile, I encourage the participants to enjoy the conference, but do not forget to dream! We have new tools to make more sense in the future, so use them wisely. In a time when we have a wealth of technology to support our human thinking, remember the words of Richard Buckminster Fuller:

You never change things by fighting the existing reality. To change something, build a new model that makes the existing model obsolete.

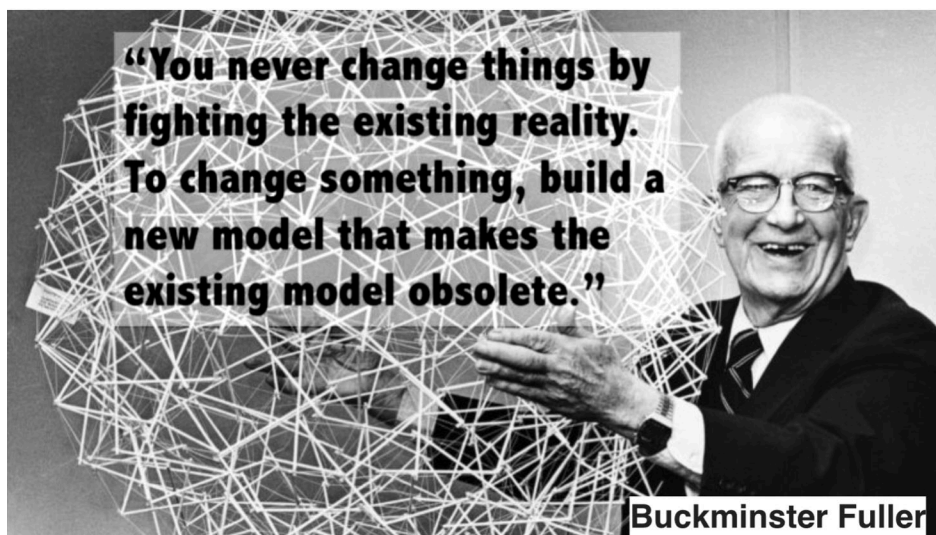


Figure 18: Build a new model that makes the existing model obsolete

Today we have technology for computation, symbol manipulation, interactive retina graphics and a verbal interface using a version of artificial intelligence. The evolution of ideas goes on into the future.

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## Plenary paper

# Making the Fundamental Theorem of Calculus Fundamental to Students' Calculus<sup>1</sup>

Patrick W. Thompson

Arizona State University, USA; [pat@pat-thompson.net](mailto:pat@pat-thompson.net)

DRAFT July 11, 2019 – to be completed

### Abstract

*I describe a calculus curriculum (from Project DIRACC)<sup>2</sup>, based in quantitative reasoning, that puts the FTC at its center. It builds from the ideas that differentials are variables whose values vary and that rate of change and accumulation are two sides of a coin. I will share results from comparisons with traditional curricula, surprising insights into different meanings of rate of change students must have at different places in the curriculum, and the central role of technology in making this approach possible.*

### Introduction

Popular U.S. calculus textbooks state the Fundamental Theorem of Calculus (FTC) as: (1) If a function  $f$  is continuous over the interval  $[a, b]$ , the function  $F$  defined as  $F(x) = \int_a^x f(t)dt$  is an antiderivative of  $f$ , and (2) If  $G$  is an antiderivative of  $f$ , then  $\int_a^b f(x)dx = G(b) - G(a)$ . To make these statements meaningful, authors build meanings of derivative, antiderivative, and definite integral prior to stating the FTC. But these textbooks' meaning for derivative is slope of a tangent line. Their meaning for integral is area of a region bounded by a curve. The net result is the FTC, in students' experience, is nothing more than a way to compute definite integrals. It adds nothing to their understanding of derivatives or integrals. Derivatives are still slopes and integrals are still areas, and the FTC says nothing about either.

The standard statement of the FTC is not true in general when integrals are areas and derivatives are slopes. The integral  $\int_a^b f(x)dx$  gives the area of a region bounded by  $x=a$ ,  $x=b$ , and  $y=f(x)$  only when  $y$  and  $x$  are ordinate and abscissa in a Cartesian coordinate system. It doesn't work in any other coordinate system. Also, in a polar coordinate system, the graph of  $y = mx + b$  is a spiral, not a line, making derivative as "slope of a tangent" unworkable. As I will clarify later, integral as accumulation from rate of change and derivative as rate of change of accumulation are not dependent upon a coordinate system.

The strong tie between integrals, derivatives, and the Cartesian coordinate system creates epistemological obstacles for students just as does tying fractions to pieces of a pie. It gives students

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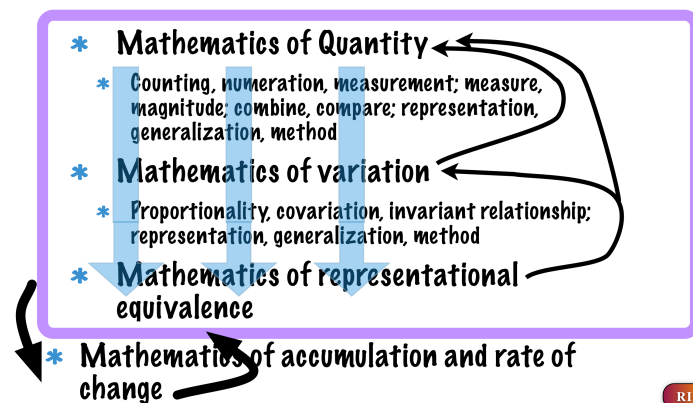
<sup>2</sup> Developing and Investigating a Rigorous Approach to Conceptual Calculus, P. Thompson (PI), F. Milner (co-PI), M. Ashbrook (co-PI). <http://patthompson.net/ThompsonCalc>

meanings for derivatives and integrals that do not generalize and, in fact, create obstacles regarding their application of integrals in non-area settings and rate of change in non-slope settings. Why, then, do textbook authors make the tie between the Cartesian coordinate system and initial ideas of derivatives and integrals so strongly? In my opinion, it is because it allows authors to pretend they are teaching ideas of calculus to students and it allows students to pretend they are learning ideas of calculus.

My goal is that students understand a calculus that is about more than lines, areas and pseudo-connections with quantitative situations. It is that students understand a calculus that arises from their reasoning about quantities and relationships among quantities. To do that, however, requires they reason about quantities and generalize their reasoning. More specifically, I wish students' comprehensions of phenomena to entail ideas of calculus they then represent symbolically.

The aim that students' calculus be rooted in their reasoning about quantities requires me to differentiate between what Les Steffe and I called students' mathematics and a mathematics of students (Steffe & Thompson, 2000). Students' mathematics is the mathematical reality they experience, which is wholly theirs and is unknowable to us in the same way dark matter is unknowable to us. Like with dark matter, the best we can do is make models that fit observations and are consistent with other models. We then use models of students' mathematics and its development (our mathematics of students) to inform our design of instruction and curricula. The calculus I share here arose in that way – from a career of creating models of students' mathematics and its potential for developing into powerful ways of thinking mathematically.

(More here later that explains the themes in the diagram below—how quantitative reasoning can provide the backbone for a mathematics of students throughout their schooling.)



## The FTC, quantitatively

To design a calculus that students might create from reasoning about their worlds quantitatively requires that we define our learning goals in these terms. For students to see the FTC as relating rates of change and accumulations students must conceptualize rate of change as a relationship between quantities whose values vary.

A particular scheme for constant rate of change of one quantity with respect to another is the earliest form of the FTC learned by some middle school students (and not possessed by some calculus



students). It entails an image of two quantities varying simultaneously so that increments in each are in constant proportion and accumulations of each are in the same proportion. I described this scheme in several earlier publications (Thompson, 1994a, 1994b; Thompson & Thompson, 1992).

Several aspects of holding this scheme warrant comment. The scheme, called FTC-E(arly), entails this imagery:

- 1) “Same proportion” means same relative size of measured quantities.
- 2) It is accumulations of quantities that vary. To a student holding the FTC-E scheme, variation implies accumulation.
- 3) Variation in an accumulation happens by its increments.
- 4) A student can envision increments happening smoothly or “chunkily” (Castillo-Garsow, Johnson, & Moore, 2013; Saldanha & Thompson, 1998; Thompson, 1994b; Thompson & Carlson, 2017). Envisioning increments happening smoothly is more advanced.

Aspect 3, variation-implies-accumulation, is a crucial component of the FTC-E scheme. Imagine a person running. If you imagine the runner’s distance scaling to a larger size, like an arrow becoming longer, you are not imagining distance accumulating. It is just becoming larger. To envision the runner’s distance accumulating one must envision the distance covered by each stride added to the runner’s distance traveled up to that stride. In this image, the runner’s accumulated distance increases with each stride. It is in this way that someone envisions variation in an accumulation happening by its increments.

Aspect 4 distinguishes, in principle, between two ways of envisioning how a quantity’s value varies. There is a distinct difference between a student imagining a runner’s distance accumulating chunkily and a student imagining it accumulating smoothly. The first student imagines the length of a completed stride being added. The second student imagines the length of a stride in progress. This distinction is important when students face the problem of modeling accumulation symbolically while taking the independent variable as varying continuously.

A far more advanced scheme, called FTC-A(dvanced), gives a later form of the FTC. A student holding the FTC-A scheme coordinates advanced schemes of variation, covariation, and constant rate of change in support of this imagery:

- 1) Two quantities vary (accumulate) smoothly and simultaneously.
- 2) They each vary in increments which themselves vary smoothly.
- 3) Increments can be small enough so, no matter how the accumulations vary, they covary through increments at an essentially constant rate of change with respect to each other.
- 4) The rate of change of the accumulations with respect to each other is the rate of change of their increments with respect to each other.

The first two aspects of FTC-A entail the idea of function as a relationship between covarying quantities. The third aspect of FTC-A entails the idea of rate of change function—a function whose values give the rate of change of an accumulation at each moment of accumulating. The fourth aspect of FTC-A is where the relationship between accumulation and rate of change is explicit. Seeing the rate of change of an increment as the rate at which an accumulation varies with respect to another



quantity is the conceptual heart of the FTC. It is the conceptual equivalent of understanding that an integral's rate of change function is the integrand of an indefinite integral.

The fourth aspect of FTC-A is nontrivial for calculus students. Figure 1 contains an item from Project DIRACC's Calculus 1 Concept Inventory given to 380 students enrolled in traditional or engineering calculus. It aims to have students consider an accumulating distance's rate of change when given information about the overall accumulation (the car's average rate of change over a four-hour period) and its rate of change over a small increment of time after that four-hour period.

A car left from San Diego heading to New York. The car's average speed for the first 4 hours of the trip was 52 mph. In the next 0.003 hours, the car had an average speed of 71 mph. Which is the best estimate of how fast the car's distance from San Diego was changing at 4 hours after leaving San Diego?

**Figure 1: FTC item from Calculus 1 Concept Inventory (© 2018 Arizona Board of Regents)**

Options presented to students are below. Comments in brackets are explanations to you.

- (a) 52 mph [miles per hour; the car's average speed over the first four hours]
- (b) 52.014 mph [the car's average speed over 4.003 hours]
- (c) 61.5 mph [the mean of 52 and 71]
- (d) 71 mph [the car's average speed over the 0.003 hours immediately after the four-hour period]
- (e) Cannot determine [There is insufficient information to answer the question]

Table 1 shows students responses to the item in Figure 1. While no option garnered a high percentage of responses, it is worth noting that 71 mph, the best approximation to the car's speed at an elapsed time of four hours, was the least popular option (13.1%). It is also worth emphasizing that 71mph is the only option consistent with FTC-A, aspect 4.

52 mph	52.014 mph	61.5 mph	<b>71 mph</b>	Cannot determine	No Answer
24.4%	17.7%	26.2%	13.1%	16.2%	2.3%

**Table 1: Responses to item in Figure 1 from 380 calculus students 13 weeks into a 15-week semester**

## DIRACC Calculus

Traditional, semester-based, university calculus in the U.S. is called Calculus 1, 2, and 3. The content of Calculus 1 traditionally covers differentiation and applications up to optimization and related rates and integration up to the FTC and elementary applications of integrals.<sup>3</sup> The content of Calculus 2 traditionally covers a potpourri of disconnected topics: advanced antidifferentiation and applications of integrals, parametric functions, sequences and series (including Taylor series), and polar coordinates. Calculus 3 covers multivariable and vector calculus.

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<sup>3</sup> This is also the content of traditional Advanced Placement Calculus BC many students take in high school.

The DIRACC curriculum covers the content of semester-based Calculus 1 and 2. However, we strived to make it grounded in quantitative reasoning and to develop ideas more coherently than traditional calculus. The coherence we aimed to create rests on what I call *Foundational Problems of Calculus*:

FP-1: You know how fast a quantity varies at every moment; you want to know how much of it there is at every moment.

FP-2: You know how much of a quantity there is at every moment; you want to know how fast it varies at every moment.

## DIRACC Calculus 1

FP-1 and FP-2 are stated at the outset of Calculus 1 and remain thematic throughout the two courses. We anticipated a dialectic between students' development of their FTC-A scheme and their work to understand and respond to FP-1 and then to FP-2.

DIRACC Calculus 1 has these central features. It:

- contains a forward-looking review of pre-calculus ideas.
- aims for students to build meaning and to reason meaningfully (reasoning based on meanings).
- emphasizes convergence, not limits.
- defines quantities so their measures are computable. The goal to compute is an organizing idea.
- aims to support students in building dynamic imagery to accompany their construction of the FTC-A scheme.

Thompson, Byerley, and Hatfield (2013) give a detailed account of the organization and content of DIRACC Calculus 1 (as of 2013). The current version is at <http://patthompson.net/ThompsonCalc>. In brief, the course evolves in four phases. The aims of each phase are:

### Phase 1: Review

This is important; most students do not understand basic ideas or hold productive imagery for them.

- Quantitative reasoning.
- Values of variables vary; strong distinctions among variables, parameters, and constants:
  - If you use a notation to represent the value of a quantity whose value varies, you are using that notation as a *variable*.
  - If you use a notation to represent the value of a quantity whose value never varies (e.g.,  $\pi$  or  $e$ ), you are using that notation as a *constant*.
  - If you use a notation to represent the value of a quantity whose value is constant within a situation, but can have different values across situations, you are using that notation as a *parameter*.
- Differentials are variables; they are the “bits” by which variables vary
- Constant rate of change is defined in terms of a relationship between differentials:  $dy = m dx$
- Large variations are made of tiny variations.
- Values of functions covary with their arguments.
- Emphasis on using function notation representationally.

- Functions defined in open form are bona fide functions just as functions defined by formulas.
- Introduce idea of “essentially equal to”.
- Conceptualize graphs, coordinate systems, and displayed graphs (the display of a statement’s graph within a coordinate system).

## Phase 2: Accumulation from rate of change

Phase 2 addresses FP-1: You know how fast a quantity varies at every moment; you want to know how much of it there is at every moment.

- 2.1 Students conceptualize constant rate of change as two quantities covarying so that variations in one are proportional to variations in the other ( $dy = m dx$ ). Indeed, we define a differential in a dependent variable only in the context that the two (say,  $dy$  and  $dx$ ) are related proportionally—they change at a constant rate with respect to each other.
- 2.2 Students represent approximate rate of change functions built from exact rate of change functions.
  - 2.2.1 An exact rate of change function  $r_f$  is a function whose values  $r_f(x)$  give the rate of change of an accumulation function  $f$  at each moment of its argument.
  - 2.2.2 An approximate rate of change function  $r$  has values  $r(x)$  which are constant over intervals of fixed length  $\Delta x$  and approximate values  $r_f(x)$ .
- 2.3 Students represent approximate net accumulation functions built from approximate rate of change functions. Approximate net accumulation accrues over each interval at a constant rate; independent variable varies smoothly. The structure of an approximate net accumulation function  $A$  is:

Assume the domain of  $x$  is partitioned in intervals of size  $\Delta x$ , starting at  $x = a$ .

For each value of  $x \geq a$ ,

$$A(x) = (\text{accumulation over complete } \Delta x\text{-intervals between } a \text{ and } x) \\ + \\ (\text{accumulation within } \Delta x\text{-interval containing current value of } x)$$

In symbols, this is

$$\text{Eq. 1 } A(x) = \left( \sum_{k=1}^{\left\lfloor \frac{x-a}{\Delta x} \right\rfloor} \left( r_f(a + (k-1)\Delta x) \Delta x \right) \right) + r(x)(x - \text{left}(x)),$$

where  $\left\lfloor \frac{x-a}{\Delta x} \right\rfloor$  is the number of complete  $\Delta x$ -intervals between  $a$  and  $x$ , and  $\text{left}(x)$  is the value of the left end of the  $\Delta x$ -interval containing the current value of  $x$ . The function called “left” is computed as

$$\text{left}(x) = a + \left\lfloor \frac{x-a}{\Delta x} \right\rfloor \Delta x$$

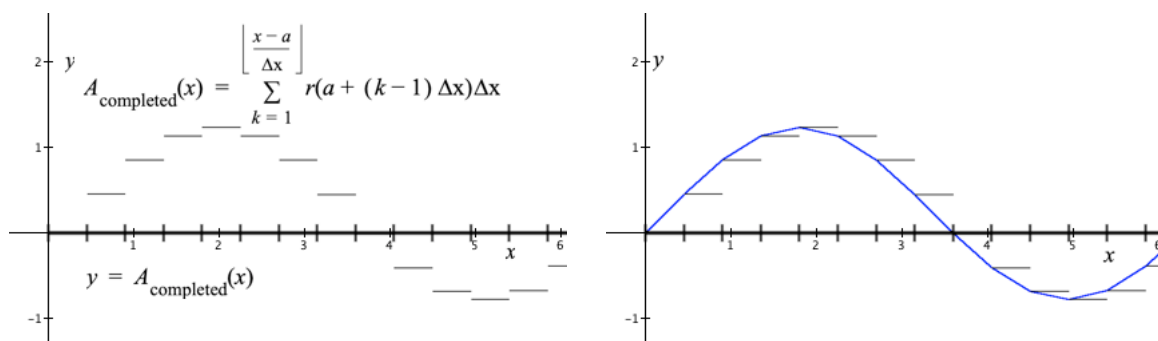
and the function  $r$  is defined as  $r(x) = r_f(\text{left}(x))$ .

It is important I point out that the construction of these function definitions is presented as a problem to students—how to actually compute values of functions we define conceptually.

I should also say that the first term in Equation 1 is not a standard Riemann sum. It has a variable upper limit, and it presumes the value of  $x$  varies smoothly. A graph of the first term alone (Figure

2, left side) is a step function—the value of  $\sum_{k=1}^{\left\lfloor \frac{x-a}{\Delta x} \right\rfloor} \left( r_f(a + (k-1)\Delta x) \right) \Delta x$  is constant for all values of  $x$  in  $[a + (k-1)\Delta x, a + k\Delta x)$  because the value of  $\left\lfloor \frac{x-a}{\Delta x} \right\rfloor$  (the number of complete  $\Delta x$ -intervals

between  $a$  and  $x$ ) is constant within the interval. Second, the graph of  $A(x)$  (Figure 2, right side) is piecewise linear because the term  $r(x)(x - \text{left}(x))$  represents the accumulation-so-far within the  $\Delta x$ -interval containing the current value of  $x$ , which varies at the constant rate  $r(x)$  within the interval. The term  $r(x)$  in  $r(x)(x - \text{left}(x))$  is analogous to option (d) in Figure 1.



**Figure 2. (Left) Graph of net approximate accumulation over completed  $\Delta x$ -intervals as the value of  $x$  varies smoothly; (Right) Graph of net approximate accumulation over completed  $\Delta x$ -intervals plus partial net accumulation within  $\Delta x$ -intervals, all as the value of  $x$  varies smoothly**

2.4 Students define exact net accumulation functions as approximate net accumulation over intervals of size  $\Delta x$  where  $\Delta x$  is so small that making it smaller gives no discernable improvement in approximations.<sup>4</sup> In DIRACC, we represent exact net accumulation in open form as

$$F(x) = \int_a^x r_f(t) dt,$$

<sup>4</sup> This approach is motivated by Cauchy's meaning of convergence, which emphasizes distance between terms instead of distance from a limit. This is how we avoid dealing with limits. We speak of convergence and terms being computationally indistinguishable from each other.

where  $r_f$  is an exact rate of change function, the value of  $x$  varies,  $t$  varies from  $a$  to  $x$ , and  $dt$  is a variable that varies through moments of the variable  $t$ .<sup>5</sup> Definite integrals are just specific values of exact accumulation functions:  $\int_a^b r_f(x)dx$  is simply  $F(b)$ .

- 2.5 We use the meaning of accumulation function to establish that if  $f$  is an (unknown) accumulation function with (known) exact rate of change  $r_f(x)$ , and known value  $f(a)$ , then total accumulation up to the value of  $x$  is

$$f(x) = f(a) + \int_a^x r_f(t)dt.$$

In words,  $f(a)$  is accumulation up to the value of  $a$  (from some unspecified reference point) while  $\int_a^x r_f(t)dt$  is net accumulation from  $a$  to  $x$ . Therefore,  $f(x)$ , accumulation up to the value of  $x$ , is  $f(a) + \int_a^x r_f(t)dt$ .

While the statement  $f(x) = f(a) + \int_a^x r_f(t)dt$  might look to you like the FTC, it is stated to students as a way to represent a complete accumulation function. It is not *stated* as a relationship between accumulation and rate of change. Rather, it *embodies* the relationship between accumulation and rate of change. It is in this way that the FTC becomes *present* in students' thinking long before it is stated for its full import.

- 2.6 “Solve” many applications using integrals by way of integrating an accumulation's rate of change function. Notice: With the aid of a computer program that “understands” integral notation, none of these applications require finding an antiderivative.

Students engage in all of 2.1 – 2.6 with the aid of a computer graphing program (called *Graphing Calculator*) that allows them to type mathematical statements in standard form to define functions, evaluate functions, and graph functions. Instructors using DIRACC at other institutions use programs like Desmos or Geogebra.

In Phase 2, we both start with concrete settings to build the mathematical ideas and emphasize applications of integrals in physical and social sciences. But none of these applications involve finding antiderivatives. Rather, students focus on conceptualizing quantities involved in situations and modeling their rate of change with respect to one another. Once they have a quantity's rate of change function, they can represent accumulation of that quantity with respect to its independent quantity using open-form integrals—which GC can graph and with which they can evaluate definite integrals. (See Appendix 1.)

### **Phase 3: Rate of change from accumulation**

Phase 3 addresses FP-2: You know how much of a quantity there is at every moment; you want to know how fast it varies at every moment.

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<sup>5</sup> The terms “exact”, “infinitesimal”, “moment”, “essentially equal to”, and “converges”, have special meanings in DIRACC. I'll explain these in my presentation.

- 3.1 Reconceive “amount” functions—functions whose values give an amount of one quantity in relation to an amount of another—as accumulation functions by envisioning the independent quantity’s value varying.
- 3.2 Connect amount functions to rate of change functions via reconceiving them as accumulation functions. For example, if  $f(x) = x^2$  gives the area of a square as its side length  $x$  varies from 0, then  $x^2 = \int_0^x r_f(t)dt$  for some rate of change function  $r_f$ .
- 3.3 Reverse the process developed in Phase 2 to construct a method for deriving exact rate of change functions defined in closed form from exact accumulation functions defined in closed form.
- 3.4 Notice this brings us full circle. In Phase 2 we started with, for example,  $r_f(x) = 2x$  and ended with  $f(x) = \int_a^x r_f(t)dt$ . In Phase 3 we started with  $f(x) = x^2$  and ended with  $r_f(x) = 2x$ . This means  $x^2 = \int_a^x 2t dt$  for some value of  $a$ . In other words, any time we find a closed-form rate of change function for a closed-form accumulation function, we’ve found a closed form definition for an open-form integral. Notice also that this is not yet the “official” FTC—we have not introduced the idea of antiderivative.
  - 3.4.1 Emphasize representational equivalence: Exact accumulation functions and exact rate of change functions can be represented in open form or closed form.
- 3.5 Standard derivations of rate of change functions from accumulation functions defined in closed form.
  - 3.5.1 Build a library of closed form definitions of open form integrals (what will later be called antiderivatives)

#### Phase 4: Applications of Derivatives

While applications play a central role in Phases 2 and 3, they are more for illustration and for students to engage in repeated reasoning with the conceptual methods being developed. Phase 4 focuses directly on applications.

- 4.1 Standard applications of rate of change functions (continuing the themes of quantitative reasoning, variation, and covariation): Related rates, optimization, graphical behaviors of functions.

#### The role of the FTC in this development

I hope it is evident that the FTC-E is at play from late in Phase 1 (differentials and constant rate of change) and the very beginning of Phase 2 (accumulation from rate of change). It segues to the FTC-A in the definition of the exact net accumulation function  $\int_a^x r_f(t)dt$  and the representation of exact accumulation as  $f(x) = f(a) + \int_a^x r_f(t)dt$ . It is restated explicitly as the traditional FTC once rate of change from accumulation is developed and the idea of antiderivative as a closed-form equivalent to an accumulation function defined in open form is crystalized.

## The role of technology in this development

DIRACC calculus would be impossible without the incorporation of the technology we use. The aim to make all definitions computable requires something to compute them. Also, defining, and having students define, functions in open form only on paper makes it improbable that they think of those functions as actually computing values of quantities.

Defining functions in open form in GC makes them “alive”, just as alive as functions defined in closed form, which in turn enhances their capacity to think of open-form definitions as representing values of quantities. Our use of technology in DIRACC is in many ways aligned with instrumentalism as described by Verillon and its adaptation in mathematics education by Artigue, Drijvers, Trouche, White, and others (Artigue, 2002; Drijvers & Gravemeijer, 2005; Trouche, 2005; White, 2019) and Heidegger’s idea of turning artifacts into tools by conceiving them as “ready at hand” in goal-oriented activity (Winograd & Flores, 1986).

There is an important aspect to our use, and students’ use, of functions defined in open form. Open form definitions are, by nature, more reflective of their meaning. Having students answer questions by defining accumulation functions and rate of change functions in open form allows them to think more clearly about what they are representing in the situation as they’ve conceived it. For example, in traditional calculus courses students spend more time finding an antiderivative in “applying” integrals than they do conceptualizing the situation in which the “problem” is embedded. Allowing students to answer questions about those same situations by identifying a rate of change and using it to define an open-form accumulation function allows them to focus clearly on conceptualizing the situation—conceptualizing quantities that compose it and relationships among them.

(Example: A student is asked to find  $\frac{d}{dx}x^{\cos(x)}$ . He writes  $\cos(x)x^{\cos(x)-1}$ . How might he check his answer? In DIRACC, he checks his answer by defining  $f(x)=x^{\cos(x)}$  and graphing both  $y=\cos(x)x^{\cos(x)-1}$  (his derived rate of change function) and  $y=\frac{f(x+0.0001)-f(x)}{0.0001}$  (the average rate of change of  $f(x)$  over every interval of length 0.0001 in its domain). When he sees dramatically different graphs (Figure 3), he concludes, because he trusts the open form definition of approximate rate of change function, that his derived rate of change function is incorrect.)

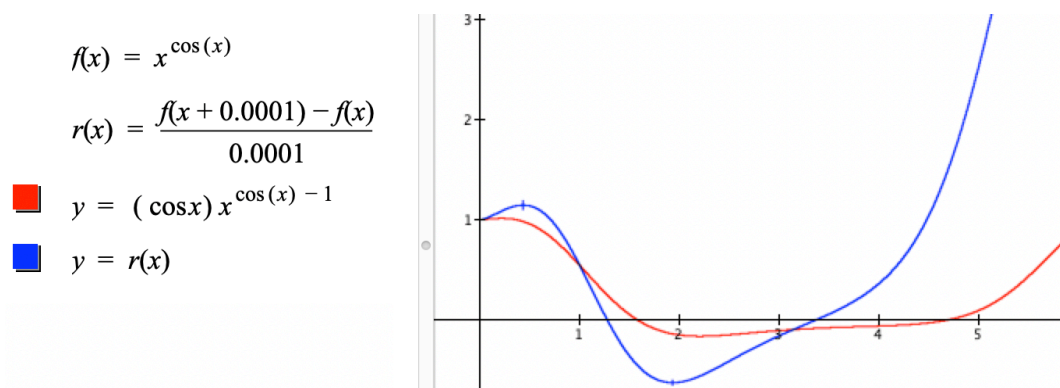


Figure 3. Graphs of student's derived and open form rate of change functions.

## Comparisons of DIRACC, traditional, and engineering Calculus 1 at ASU

The mathematics department at ASU, like many U.S. universities, has different “flavors” of calculus: Calculus for business majors, life science majors, mathematics and science majors, and engineering majors. Business and life science calculus focuses on applications and contain very little concept development. Mathematics/Science and engineering calculus are largely identical in topics, but engineering calculus holds class three hours per week while mathematics/science calculus holds class four hours per week. Within math/science calculus we offered both traditional and DIRACC calculus for three years to compare them; DIRACC is now ASU’s calculus for math/science majors.

In Fall 2015 we compared students’ performance in DIRACC and traditional calculus using an 11-item pretest at the beginning of the semester, then including the pretest within students’ final examination as a posttest. A committee of five people, two teaching traditional Calculus 1, two teaching DIRACC Calculus 1, and the department’s director of STEM education, constructed the pretest/posttest. No item was included without unanimous agreement that it assessed a central idea in calculus and addressed it acceptably (see Appendix 2).

Table 2 contains results from the pretest and posttest. The pooled  $t$ -test for posttest comparison is  $t = 11.853$ ,  $p < 0.0001$ .

	<u>PreTest</u>		<u>PostTest</u>	
	<u>Mean</u>	<u>StdDev</u>	<u>Mean</u>	<u>StdDev</u>
Traditional $n = 248$	3.18	1.53	4.89	2.45
DIRACC $n = 149$	2.98		7.90	

**Table 2. Pre-Post Comparison of Traditional and DIRACC Calculus 1. Possible score = 11.**

The construction of two “concept inventories” for semester-based Calculus 1 and 2 was part of the National Science Foundation’s charge to DIRACC in their funding award. The Calculus 1 Concept Inventory (C1CI) assessed students’ understandings in the areas under the headings Variation and covariation, Function, Modeling/Quantitative Reasoning, Structure sense, Rate of change, Accumulation, and Fundamental Theorem of Calculus. Examples from each area are presented in Appendix 3. Table 3 contains Scheffé comparisons among DIRACC, traditional, and engineering calculus students’ scores on the C1CI near the end of their semester. Interpret  $p$ -values hesitantly. Students taking the C1CI were recruited from each program’s courses with an inducement of \$50 each to participate, so sampling was subject to unknown sources of bias.



	<i>Difference</i>	<i>Std Error</i>	<i>p-Value</i>
<i>DIRACC vs Engineering</i>	3.95	0.90	0.001
<i>DIRACC vs Traditional</i>	2.26	1.10	0.13
<i>Traditional vs Engineering</i>	1.68	0.90	0.18

**Table 3. Scheffé comparisons after significant ANOVA for Calculus 1 Concept Inventory.**

### **Department-required Derivatives Test**

For as long as anyone in ASU's mathematics department can remember, the department requires Calculus 1 students to pass a derivatives mastery test to pass the course. Students get three attempts to pass the test. We were unable to get pass rates from traditional and engineering instructors who declined to share them. The department's undergraduate director did share with us that the overall DIRACC pass rate was approximately the same as other programs, and our pass rate on first attempt was higher.

### **DIRACC Calculus 2**

The above description of the DIRACC Calculus 1 is necessarily sparse. I'd like to spend more time describing and exemplifying how the themes developed in Calculus 1 lay a foundation for a coherent development of ideas in Calculus 2—which typically stand in splendid isolation from Calculus 1 and from each other.

As I mentioned, traditional Calculus 2 in the U.S. is a potpourri of disconnected topics. My challenge for Calculus 2 was to reconceptualize these topics so they are coherent with ideas of accumulation from rate of change and rate of change from accumulation and coherent with each other. The idea of having students conceptualize differentials as variables was key to developing a coherent Calculus 2.

The DIRACC approach to quantifying regions bounded by graphs and regions within solids of revolution might clarify the power of conceiving differentials as variables. I share these examples instead of equally interesting examples from the sciences because so many people ask how area and volume are treated in the DIRACC curriculum.

### **Quantifying Regions in the Plane Bounded by Graphs**

Computing areas of regions bounded by graphs in the Cartesian coordinate system is the traditional context in which integrals are developed. The idea is to sum areas of rectangles with bases on the horizontal axis and one vertex on the function's graph. This development creates a number of cognitive difficulties for students, including: (1) Area is always positive, so regions above and below the graph must be computed separately. (2) Only definite integrals are addressed; the idea of integral as a function is absent. (3) Integral as area makes students wonder how, for example, computing an area can answer a question about distance, force, or any non-area quantity. (4) Connections between integrals and rate of change are absent.

On the other hand, area of a bounded region is a quantity. It should be possible to apply the DIRACC approach of accumulation from rate of change to the quantification of a planar region bounded by graphs and a spatial region bounded by a surface of revolution.

Suppose a rectangle has height 3 cm and varying base length of  $x$  cm. The rate at which the rectangle's area changes with respect to its base length is  $3 \text{ cm}^2/\text{cm}$ . We emphasize to students that even though the rate of change of area with respect to base length has the same *value* as the rectangle's height, the rectangle's height is not the rate at which area changes with respect to base length. The rectangle's height is 3 cm. The rectangle's rate of change of area with respect to base length is  $3 \text{ cm}^2/\text{cm}$ . They are different quantities that have the same value.

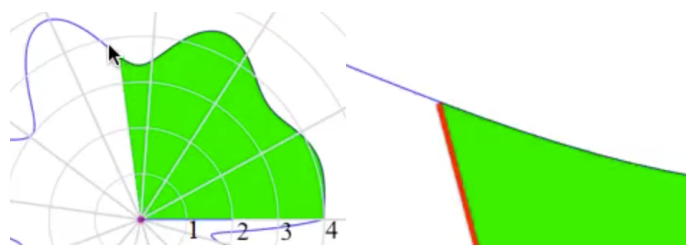
More generally, the differential  $dA$  of bounded area in the Cartesian coordinate system is the area of a rectangle of height  $f(x)$  and base length  $dx$ . When the height of a rectangle is  $f(x)$  and its base length varies by  $dx$ , its rate of change of area with respect to  $x$  has the same value as  $f(x)$  (see Figure 4).



**Figure 4. The area of a rectangle with constant height and varying base has a rate of change of area with respect to its base equal in value to the measure of its height.**

In DIRACC Calculus 1, we developed that when any quantity has a rate of change function  $r_f(x)$ , net accumulation in  $f$  as the value of  $t$  varies from  $a$  to  $x$  is  $A_f(x) = \int_a^x r_f(t) dt$ . Therefore, net accumulated area bounded by the graph of  $y = f(x)$ ,  $x = a$ , and  $x = b$  is  $\int_a^b f(t) dt$ . However, the idea of “net accumulated area” must mean the quantity is *signed* area. Over intervals where  $f(x)$  is negative, area accumulates at a negative rate of change, so net change in area will have a negative measure.

The fact that the value of  $f(x)$  is the rate of change of signed area with respect to  $x$  depends on the graph being displayed in the Cartesian coordinate system. In a polar coordinate system, the rate of change of area bounded by the graph of  $r = f(\theta)$  as the value of  $\theta$  varies is not  $f(\theta)$ . The differential for the area of a bounded region in a polar coordinate system is a sector of a circle (see Figure 5).



**Figure 5. The differential of area bounded by a graph in polar coordinates is area of a sector of a circle with radius  $f(\theta)$  and angle measure  $d\theta$ .**

Since  $dA$ , the differential in area, changes at a constant rate with respect to  $d\theta$ ,  $dA = m d\theta$ . Moreover,  $dA = \pi \frac{r^2}{2}$  when  $d\theta = \pi$ . So, in polar coordinates,  $dA = \frac{r^2}{2} d\theta$ . The exact rate of change function for area bounded by a graph in polar coordinates is therefore  $r_A(\theta) = \frac{f(\theta)^2}{2}$  and accumulation of unsigned area bounded by a graph in the polar coordinate system is  $\int_a^\theta \frac{f(t)^2}{2} dt$ .

Unsigned area is not the same as net accumulated area. Net accumulated area is a signed quantity—it must have a negative value when the value of the function is negative. However,  $f(t)^2$  is always non-negative. We can adjust to this by multiplying the integrand by the sign of the function's value, giving

$$A(a, \theta) = \int_a^\theta \left( \operatorname{sgn}(f(t)) \frac{f(t)^2}{2} \right) dt.$$

### Quantifying Regions bounded by surfaces of revolution

In DIRACC we speak of surfaces of revolution and regions they bound rather than speaking of solids of revolution. This decision arose from two sources—our commitment to crafting a dynamic calculus and interviews with students studying traditional developments of volumes of solids of revolution.

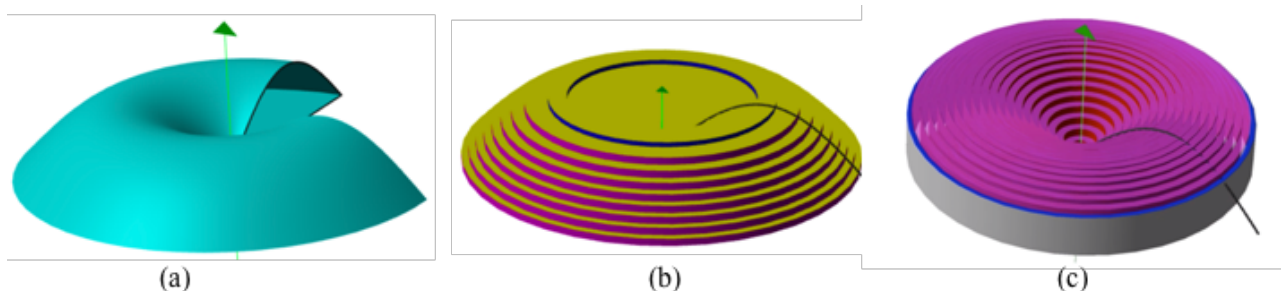
In traditional developments, students see textbooks proposing to generate solids by revolving a bounded region in the plane around an axis, then slicing the solid into pieces whose volumes they approximate with disks, washers, or shells. Sometimes textbooks propose to first approximate the region with rectangles before revolving so that the revolution itself generates disks, washers, or shells.

I conducted a Calculus 2 exploratory design experiment in Spring 2017. What this means is I wrote just-in-time modules for upcoming topics, asked probing questions during class, and interviewed students during office visits and volunteers outside of class.

My first pass at solids of revolution resembled traditional approaches with a small difference—treating slices as differentials. What I quickly noticed was that students exhibited what every Calculus 2 instructor witnesses – students were confused about whether to use disks, washers, or shells in setting up their integrals.

Further probing led me to conclude the root of their problem was in their conceptualization of the solid. It was entirely unnatural to think of anything varying within it. It also became evident that students did not see a connection between a method they contemplated and the integral's independent variable. I pondered how to address the two issues simultaneously.

The first modification was to change the metaphor for solids of revolution. Instead of revolving a region to generate a solid, I spoke of revolving the graph of  $y = f(x)$  about the  $y$ -axis or  $x$ -axis to generate a surface (Figure 6a), pointing out that the surface encloses an empty region in space. I then offered the metaphor of “filling” the enclosed region, asking about ways to fill it so we can quantify the region (compute an approximation to its volume; Figure 6b and Figure 6c).



**Figure 6. Quantifying a region enclosed by a surface of revolution. Method (b) fills the region with cylinders having constant base and varying height. Method (c) fills the region with cylinders having constant height and varying base. Volume of a blue-edged cylinder is the differential in volume for the respective method of accumulation.**

There are three benefits of this approach. First, students distinguish between creating a region in space and making a solid. They make a solid by filling an empty shell after they revolve a graph to make it. Second, they must decide explicitly which variable (original function's dependent or independent variable) will be their independent variable of accumulation. They can choose either, and their choice will determine the kind of cylinders they will use to fill the empty region, which in turn will determine the rate of change of accumulating volume with respect to the accumulation's independent variable. Third, they only need to think of two kinds of cylinders—cylinders having constant base and varying height, or cylinders having constant height and varying base.<sup>6</sup>

In Figure 6b,  $y$  is the independent variable of accumulation and the differential of approximate accumulating volume is  $dV = (\text{area of cylinder's base})dy$ , which means approximate volume's rate of change function has the same numerical value as the area of the cylinder's base. In Figure 6c, the differential in volume is  $dV = 2\pi xf(x)dx$ , which means the accumulating volume's rate of change function is  $r_v(x) = 2\pi xf(x)$  and volume is  $V(a, x) = \int_a^x r_v(t)dt$ .

Figure 6 (b and c) also illustrates how this approach helps students clarify decisions they must make about which variable to use as accumulating volume's independent variable. If a student chooses (b), taking  $y$  as the accumulation volume's independent variable, she will be forced to use  $f^{-1}$  to compute inner and outer radii of her cylinders. In this instance, students can avoid this difficulty by using  $x$  as accumulating volume's independent variable. The point is that students consider which variable to take as their independent variable of accumulation apart from which variable is the original function's independent variable. In the traditional approach, students were unaware that volumes they tried to compute had an independent variable.

### Motive for these examples

I shared this disquisition on signed area in Cartesian and polar coordinate systems and volume of regions bounded by a surface of revolution to illustrate the coherence gained across topics by the

<sup>6</sup> We develop a general definition of right cylinder as any geometric figure having congruent top and bottom lying in parallel planes and having sides perpendicular to both.

thematic attention to quantities and their variation and by casting differentials as variables. Standard developments of integrals and derivatives based in static geometric arguments presume a timeless, unchanging world from which mathematics derives meaning—a world counter to students’ lived experience. This presumption also leads to standard calculus having a strong focus on geometric interpretations of definite integrals as areas and derivatives as slopes of tangents at a point.

I hasten to add that the mathematics students create through school (at least in the U.S.) is their accommodation to a mathematics presented as if nothing varies. Variables’ values change by substituting one number for another. Students having this conception of variable cannot use letters to represent quantities’ values in dynamic settings, which leads to students’ evolving beliefs that mathematics is about symbolic rituals. I address students’ felt conflicts between their school mathematics and DIRACC calculus in the next section.

### **Approximating Integrals and Polynomial Approximations**

(I’m unsure whether to include this discussion. While it extends the other two examples in illustrating the coherence of the DIRACC approach, the discussion might add too much to the paper’s length.)

### **Students’ Reactions to DIRACC Calculus**

### **Related Work (to be completed)**

(Rosenthal, 1992)(Strang, 1990, 1991)(Macula, 1995) in some ways foreshadow what I’ve said about the FTC-E and FTC-A schemes. However, they (1) focus on finding areas under curves, (2) are unconcerned with students’ images of variation, and (3) their primary goal is the second form of the FTC  $[F(b)-F(a)]$ . Moreover, Strang emphasizes change in area by combining static bits

$$\left(F(x_n) - F(x_{n-1})\right) + \left(F(x_{n-1}) - F(x_{n-2})\right) + \cdots + \left(F(x_2) - F(x_1)\right) = F(x_n) - F(x_1)$$

rather than a summation of terms  $f(x_i)\Delta x_i$  – the quantities made by a rate of change over an interval of change.

An explosion of focus on and research on accumulation functions in calculus:

(Bressoud, 2009, 2011)(Sealey, 2014; Swidan, 2019; Swidan & Naftaliev, 2019; Swidan & Yerushalmy, 2015) (Kouropatov & Dreyfus, 2014). Point out that they omit the idea of rate of change in their meaning of accumulation.

Comment on Ely’s (Ely, 2010, 2017) notions of infinitesimals and “smooth continuous” and difficulty with building rate of change conceptually from them.

Image for presentation – change, with prior changes “evaporating” upon next change.

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## Appendix I

### Assignment Sheet for Students: Measuring Energy Consumption<sup>7</sup>

Energy is commonly measured in units of Joules (J), which is the work done by a force of one Newton when its point of application moves one meter in the direction of the force. The rate at which energy is used is known as power. Power is measured in Joules/sec (1 Joule/sec is called a Watt). It is also common to measure an amount of energy in kilowatt-hours (kWh), which is the amount of energy used when it is used at the rate of 1000 Joules/sec (1000 Watts) for one hour. A megawatt hour is the amount of energy used when it is used at the rate of one million Watts (one million Joules/sec) for one hour.

Suppose a city consumes electrical energy on a given day at an approximate rate of  $r(t)$ , where  $t$  is a number of hours since midnight and  $r$  is defined, in megawatts, as

$$r(t) = 400 - 300 \cos\left(\frac{\pi}{12}(t-5)\right), 0 \leq t$$

- Fill in the blanks in 5 different ways to make this statement true:  
3 Megawatt-hours = The energy consumed at a rate of \_\_\_\_\_ Joules/sec for \_\_\_\_\_ hours.
- Why is it sensible that  $\frac{\pi}{12}(t-5)$  is the argument to cosine in this model of the city's rate of electrical energy consumption? (Examine the graph of  $r$ .)
- Define a function that gives the electrical energy this city will have consumed  $x$  hours after midnight on a given day. Explain how your function produces a value that is in the desired units. Does the value of  $x$  have a necessary upper bound?
- Approximately how much electrical energy does this city use in a typical day? In a typical week? In a typical month? In a typical year? Be sure to state quantity's units.
- This city's electric utility company charges for electricity at the rate of \$0.13 per kilowatt-hour for electricity used between 7:00a and 5:00p, and at the rate of \$0.07 per kilowatt-hour otherwise. What is this city's electrical bill for one day?
- Burning 1 kg of coal produces about 450 kWh of energy. How many kg of coal are required to meet the energy needs of the city for one day? One year?
- A wind turbine normally generates electricity at a rate of 200 kW. Approximately how many wind turbines would be required to meet the needs of this city for one day? One year?
- A medium-sized household typically uses about 300-500 kilowatt-hours of electrical energy in a month during the fall. Residential consumption is typically about 30% of a major city's total

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<sup>7</sup> Adapted from Briggs & Cochran, Chapter 6.1, Problem 57.



electrical energy consumption in that same period. Approximately how large is this city's population?

## Partial Student Response using GC

☒  $r(t) = 400 - 300 \cos\left(\frac{\pi}{12}(t - 5)\right)$

This is the rate at which amount of electrical energy that the city has used is changing  $t$  hours after midnight on a particular day. The value of  $r$  is a number of megawatts (1 million Joules/sec).

☒  $E(x) = \int_0^x r(t) dt$

(c)  $E$  is a function that gives the amount of electrical energy that the city uses (in megawatt-hours) from midnight to  $x$  hours after midnight on a particular day. The value of  $r$  is in megawatts and the value of  $dt$  is in hours, so the value of  $r(t)dt$  for a value of  $t$  and a really small value of  $dt$  is in megawatt-hours. So the value of  $E$  is in megawatt-hours.

The value of  $x$  can be any positive number (assuming the model remains applicable). For example, because the model is periodic,  $E(48)$  would be the electrical energy that the city used in two days from the beginning of the first day.

☒  $E(24) = 9600$

(d)  $E(24)$  gives the number of megawatt hours that the city uses in one day (24 hours from midnight on a particular day). According to this model, the city would use 9600 megawatt-hours of energy in one day. Energy used in a week would be  $E(7*24)$  megawatt-hours, in a typical (30-day) month the city would use  $E(30*24)$  megawatt-hours and in a (365-day) year it would use  $E(365.25*24)$  megawatt-hours of electrical energy.

☒  $P_1 = 0.13 \times 10^3, P_2 = 0.07 \times 10^3$

$P_1$  and  $P_2$  are the prices of electricity per megawatt-hour that the electrical company charges during the two different periods of the day.

☒  $P_1 (E(17) - E(7)) + P_2 (E(7) + E(24) - E(17)) = 946377$

(e)  $E(17) - E(7)$  is the amount of electrical energy that the city uses between 7am and 5pm on a particular day.  $E(7)$  is the amount of energy used between midnight and 7am, and  $E(24) - E(17)$  is the amount of energy that the city uses between 5pm and midnight. The sum of  $E(7)$  and  $E(24) - E(17)$  is the amount of energy the city uses at a price of  $P_2$  dollars per megawatt-hour, and  $E(17) - E(7)$  is the amount of energy the city uses at a price of  $P_1$  dollars per megawatt-hour. According to this model, this city spends about \$946,000 per day for electricity.

## Challenge

It is reasonable that this city uses less electricity on weekends than on weekdays.

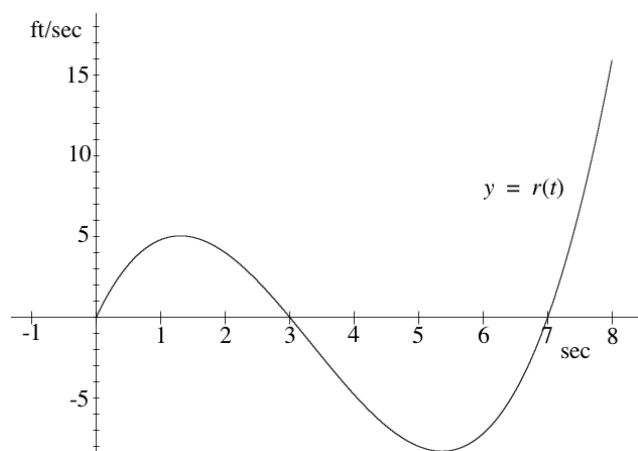
- Define a function that, in your estimation, reasonably models the rate of electricity usage on weekends. Explain your definition.
- Redefine the overall rate function so it models rate of change of energy usage for any (real) number of days since the beginning of some Monday.
- Give new answers to the original questions.

## Appendix 2

### Pretest/Posttest for Comparing DIRACC and Traditional Calculus 1

1. The function  $r$  is the *rate of change function* with respect to time for a particle's displacement from its initial position while it moves in a straight line. The graph of  $y = r(t)$  is given to the right.

The function  $s$  is the particle's *displacement function*. Its values (measured in feet) give the particle's displacement from its initial position  $t$  seconds after starting. At what time, approximately, during the first 7.5 seconds does  $s(t)$  have its smallest value?



- A. 1.2 sec  
 B. 3 sec  
 C. 5.4 sec  
 D. 7 sec  
 E. None of the above

2. A company produces different sized smartphones with rectangular screens. The screen's dimensions are  $w$  and  $h$ , where the height ( $h$ ) is half the width ( $w$ ) for all sizes of smartphones. Which of the following functions represents any screen's diagonal length as a function of its width?

A.  $L(w) = \frac{\sqrt{5}w}{2}$

B.  $L(w) = wh$

C.  $L(w) = \frac{1}{2}w^2$

D.  $L(w) = \sqrt{w^2 + h^2}$

E. None of the above

3. The Trans-Port Company manufactures containers of various dimensions, with heights  $x$  up to 4.5 yards. The volume of their containers of height  $x$  is given by the function  $g$ , where  $g(x) = 4x^3 - 50x^2 + 144x$  is measured in cubic yards. If the height of the container is increased from 1.5 yards to 2 yards, what is the corresponding change in the container's volume, in cubic yards?

A.  $(2 - 1.5)^3$

B.  $g(2 - 1.5)$

C.  $2^3 - 1.5^3$

D.  $g(2) - g(1.5)$

E.  $g(1.5) - g(2)$

4. When a rocket is launched, its speed increases continually until its booster engine separates from the second stage. During the time it is continually speeding up, the rocket is never moving at a constant speed. What, then, would it mean physically to say that at *precisely* 2.15823 seconds after launch the rocket is traveling at *precisely* 183.8964 miles per hour?
- A. 183.8964 is the limit of a difference quotient as time approaches 2.15823 seconds.
- B. If you were to freeze time at 2.15823 seconds after launch, the rocket's speedometer would point at 183.8964 miles per hour.
- C. The rocket traveled at the speed of 183.8964 miles per hour for the first 2.15823 seconds of its flight.
- D. The rocket's speed over the time interval of 2.15822 seconds to 2.15824 seconds after launch is essentially 183.8964 miles per hour.
- E. None of the above is an acceptable meaning for the statement that the rocket was going precisely 183.8964 miles per hour 2.15823 seconds after launch.
5. The table below gives information about functions  $f$  and  $g$ . Let  $h$  be defined as  $h(x) = f(g(x))$ . What is the rate of change of  $h$  at  $x = 4$ ?

	$x$			
	1	2	3	4
$f(x)$	20	23	18	14
<i>Rate of change of <math>f</math> at <math>x</math></i>	7	-3	-7	-2
$g(x)$	-10	-11	-4	2
<i>Rate of change of <math>g</math> at <math>x</math></i>	-0.5	2	4	3

- A. -2      B. -3      C. -6      D. -7      E. -9

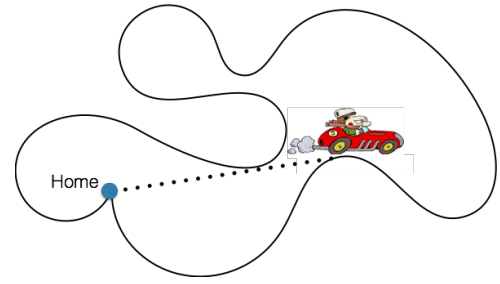
6. What is the primary focus of calculus?
- A. Properties of graphs, mainly slopes and areas
- B. Finding values of derivatives and integrals
- C. Modeling and analyzing how quantities vary together
- D. Learning complex operations with symbols and numbers to improve cognition
- E. Finding limits

For questions 7 – 10: Let  $F(x) = \int_a^x f(t)dt$

7. What does  $f$  represent?
  - A. A distance function with respect to time
  - B. A small change in a quantity
  - C. A rate of change function for some quantity
  - D. A total amount of some quantity
  - E. None of the above
8. What does  $f(t)dt$  represent?
  - A. A distance function with respect to time
  - B. A small change in a quantity
  - C. A rate of change function for some quantity
  - D. A total amount of some quantity
  - E. None of the above
9. What does  $F$  represent?
  - A. A distance function with respect to time
  - B. A small change in a quantity
  - C. A rate of change function for some quantity
  - D. A total amount of some quantity
  - E. None of the above
10. What does  $t$  represent in the expression  $f(t)$ ?
  - A. Time
  - B. The value half way between  $a$  and  $x$
  - C. A variable that varies from  $a$  to  $x$
  - D. Nothing, it is a dummy variable
  - E. None of the above

11. Bob traveled in his car at a constant speed along a complicated loop, beginning and ending at his home. What must be true about the rate of change of the car's straight-line distance from home with respect to time at the moment it is farthest from home?

- A. The rate of change will be largest at the moment the car is farthest from home.
- B. The rate of change will change from negative to positive at the moment the car is farthest from home.
- C. The rate of change will be zero at the moment the car is farthest from home.
- D. The rate of change will be smallest at the moment the car is farthest from home.
- E. None of the above.

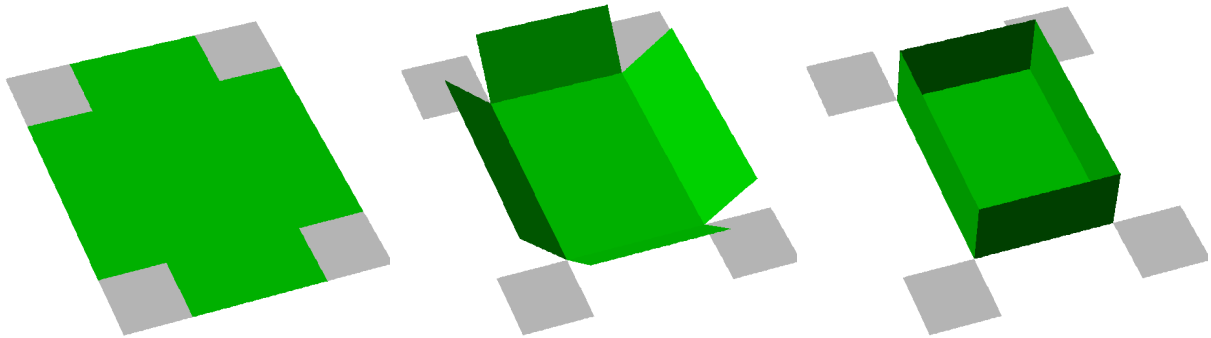


## Appendix 3

Sample C1CI Items (all © 2018 Arizona Board of Regents)

### Variation and Covariation

1. You have an  $x$  cm by  $y$  cm rectangular sheet of cardboard. You can fold the sheet into a box by first cutting squares with side lengths  $a$  cm from each of the four corners. Which of  $a$ ,  $x$ , and  $y$  have values that vary when you think of finding the box with the largest possible volume?



- a)  $a$
- b)  $x$
- c)  $y$
- d)  $x$  and  $y$
- e)  $a$ ,  $x$ , and  $y$

## Function

2. A function  $f$  converts weight in pounds (at a particular location on earth) to the equivalent mass in kilograms. Another function  $g$  determines the volume of a certain liquid in liters as a function of the total mass of the liquid in kilograms.

Given a certain volume  $x$  of this liquid in liters, which of the following is the weight of the liquid in pounds?

- a)  $f^{-1}(g(x))$
- b)  $f(g^{-1}(x))$
- c)  $g^{-1}(f(x))$
- d)  $g(f(x))$
- e)  $f^{-1}(g^{-1}(x))$

## Modeling/Quantitative Reasoning

3. The Trans-Port Company manufactures containers of various dimensions, the tallest being 3.5 yards tall. The volume of a container depends on its height;  $g(x) = 4x^3 - 50x^2 + 144x$  is the volume (in cubic yards) of a container with height  $x$  yards. If the height of the container is increased from 1.5 yards to 2 yards, what is the corresponding change in the container's volume, in cubic yards?

- a)  $(2 - 1.5)^3$
- b)  $g(2 - 1.5)$
- c)  $2^3 - 1.5^3$
- d)  $g(2) - g(1.5)$
- e)  $g(1.5) - g(2)$

## Structure Sense

4. For the following function, which differentiation rule applies to the expression as a whole?

$$((f(s)/g(s)+k(s))^3\sqrt{h(s)})$$

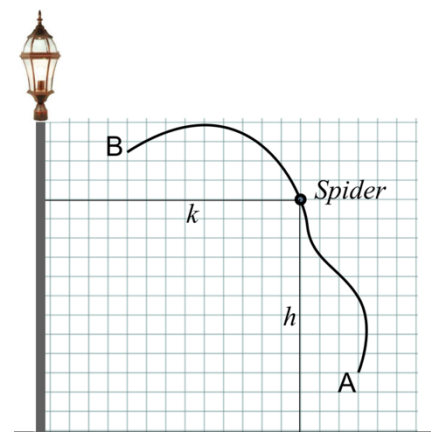
- a) Chain Rule
- b) Power Rule
- c) Product Rule
- d) Quotient Rule
- e) Sum Rule

## Rate of Change

5. At one end of a brick wall is a vertical light pole. A spider walks on the wall from point A to B along the path shown. The number of feet the spider is above the ground ( $h$ ) and the number of feet the spider is to the right of the light pole ( $k$ ) both vary as the spider walks the path.

Estimate the rate of change of  $k$  with respect to  $h$  at the moment shown in the illustration.

- a) 1
- b) 2
- c) -2
- d) -1/2
- e) Not enough information to make an estimate



## Accumulation

6. On Mars, an astronaut dropped his watch from a cliff. Its speed at every moment was  $w(t)$  meters per second, where  $t$  is the number of seconds after the watch was released. Which expression gives the *best* estimate for the distance the watch fell from 8 to 8.04 seconds after being released?

- a)  $w'(t) \cdot (8.04 - 8)$
- b)  $w(8)(8.04 - 8)$
- c)  $\frac{w(8.04) - w(8)}{8.04 - 8}$
- d)  $w(t) + w(t + 0.02)(0.02)$
- e)  $w(8)(0.02) + w(8.02)(0.02)$



## Fundamental Theorem of Calculus

7. Given a differentiable function  $k$ , for what values of  $a$  is  $\int_a^x k'(t) dt = k(x)$  for all values of  $x$ ?

All values of  $a$  such that...

- a)  $k'(t) = k(a)$
- b)  $x = a$
- c)  $k(a) = 0$
- d)  $k'(t) = k(x)$
- e)  $a = 0$

## Plenary panel

### **From Newton's first to second law: How can curriculum, pedagogy and assessment celebrate a more dynamic experience of calculus?**

Alejandro S. González-Martín<sup>1</sup>, Vilma Mesa<sup>2</sup>, John Monaghan<sup>3</sup> and Elena Nardi<sup>4</sup> (Moderator)

<sup>1</sup>Université de Montréal, Canada; [a.gonzalez-martin@umontreal.ca](mailto:a.gonzalez-martin@umontreal.ca)

<sup>2</sup>University of Michigan, USA; [vmesa@umich.edu](mailto:vmesa@umich.edu)

<sup>3</sup> University in Agder, Norway and University of Leeds, UK; [john.monaghan@uia.no](mailto:john.monaghan@uia.no)

<sup>4</sup>University of East Anglia, UK; [e.nardi@uea.ac.uk](mailto:e.nardi@uea.ac.uk)

#### **Outline**

Calculus is a formidable toolbox for the study of change. Yet, at a time when digital technologies provide the capacity to create and celebrate dynamic experiences of calculus, institutional and other challenges may impede embracing this capacity in its curriculum, pedagogy and assessment. In many high stakes assessment systems, for example, coursework, formative testing and closed-book examination tasks seem to be stuck in the pre-digital age. Systemic inertia seems to manifest itself in other ways too. Calculus is needed in different shapes and forms in the different disciplines and professions; yet, it is typically introduced to those who study disciplines other than mathematics without due regard to the needs of the discipline. And, even though students often find calculus challenging and irrelevant—and, consequently, may disengage—it is still offered, unchanged, to them devoid of the *raison d'être* for its uses in their disciplines. The panelists first, and briefly, shared their experiences in the study and design of curriculum and assessment materials for calculus. Then, in the second and longer part of the panel session, through examples from those experiences, they mapped out one possible way of fostering change: designing tasks—for classroom activity as well as assessment—that convey important meanings of calculus, are accessible, celebrate its dynamism, and are tailored to the needs of students in various disciplines who will soon enter diverse worlds of work.

Following a brief introduction by the moderator, each panelist addressed aforementioned current state of affairs ("*Inertia*"). A first round of audience interventions, collated before the event, as well as on the spot possibly via a shared link, followed. The moderator then briefly pulled together the threads of the discussion so far and a second round of contributions from each panelist followed on how to foster change ("*Force and acceleration*"). A further round of audience interventions will ensue and the session closed with a final comment from each panelist and the moderator.

# Teachers' choices of digital approaches to upper secondary calculus

Henrik Bang<sup>1</sup>, Niels Grønbaek<sup>2</sup> and Claus Larsen<sup>1</sup>

<sup>1</sup>Christianshavns Gymnasium, Denmark; [hb@cg-gym.dk](mailto:hb@cg-gym.dk) and [cl@cg-gym.dk](mailto:cl@cg-gym.dk)

<sup>2</sup>University of Copenhagen, Denmark; [gronbaek@math.ku.dk](mailto:gronbaek@math.ku.dk)

## Background

Danish mathematical education has put forward a rather extensive use of digital technologies over the last 10 – 15 years to enhance mathematical possibilities but with the possible adverse effect of diluting core mathematical insight. Instigated and supervised by CMU<sup>i</sup>, 95 upper secondary mathematics teachers have participated in an enterprise<sup>ii</sup> to counter this risk by explicitly using mathematical software, foremost CAS, to develop competences within mathematical intuition, logical methods and understanding of mathematical entities. Participating teachers' skills in use of mathematical software ranged from novice to expert. Their projects reflect what ordinary teachers do if they are given the opportunity to follow their own ideas. Our analysis of the diversity of approaches points to three important circumstances: A design dilemma between anticipated learning paths and students' actual activities; the importance of teachers' strategic choices of 'outsourcing' tasks to mathematical software; emphasis on 'instrumental genesis'.

## The slider trap

We use the term 'slider trap' for the design dilemma. It involves the two other circumstances mentioned in the beginning, so we concentrate on the slider trap. It takes the form of a discrepancy between the teacher's design of students' computer activities based on a hypothetical perception of students learning trajectories through problems that they must overcome to unfold the learning potential and the students' actual activities - ending up to demonstrate that there is too little opportunity for learning in the design. A digital environment often amplifies this general dilemma.

One CMU project (Lauritzen, 2015) attracted our attention to the slider trap. It aimed at improving students' understanding of the *3-step-method* (cf. below) through a hands-on CAS experience in the hope of ending previous years' teaching frustration. The teacher (L) concludes his project report (our transl.): Students asked fewer questions when I [later] lectured on the 3-step-method, but a student's work at the blackboard was as defective as usual. We provide excerpts from an interview:

- |              |   |
|--------------|---|
| Interviewer: | Did you pose problems [for the CAS-version of 3-step-method]?   |
| Teacher:     | No, that is too complicated. ... We can easily arrange it and [we think that] now they can surely see it, but they cannot see a bit. There goes pedagogy. |
| Interviewer: | Is it an animation [of 3-step-method]?  |
| Teacher:     | [Yes,] instead of you in the old days sketched a secant on the blackboard and then a new secant.  |
| Interviewer: | When you say 'proof of 3-step-method', is it then proof for specific functions?   |

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<sup>i</sup> Centre of Computer-assisted Mathematics Instruction, Dept. Math. Sci., University of Copenhagen.

<sup>ii</sup> 2015 – 2017, funded by The Danish Industry Foundation

- Teacher: As standard we use x squared. There are many partial elements.
- Interviewer: What about square root x?
- Teacher: The idea is that the students can do the full 3-step-method without using so much energy on it ... All they have to do is to key in a new function and then they can see the steps. ... I had not expected them to take a function so tricky.

Even though the learning outcome appears unimproved, L expresses that the project has given him much deeper insight into the learning circumstances for the three-step-method.

The teacher side of the dilemma concerns a priori subject matter analysis, choices of learning tasks and environment for corresponding student activities. The teacher's strategy w.r.t. the fundamental outsourcing question of balancing task efficiency with control of outcome (Bang, Grønbaek, & Larsen, 2017) is particularly important. Our discourse on students' computer activities will draw on concepts from the theory of *instrumental genesis* (Trouche, 2014; Drijvers & Gravemeijer, 2005): Briefly, the computer is an input-output device with constraints and potentials, an artefact. Instrumental genesis is a synthesis of the artefact and the user's methods and knowledge to the effect of meaningful accomplishment of purposeful mathematical tasks. It shows as *instrumented action schemes* composed of *technical and conceptual elements* manipulated by *instrumented techniques*, whose value have dual character: *pragmatic* by providing results and *epistemic* by enhancing the user's knowledge of the objects for the techniques (Artigue, 2002).

A *CAS-app* is a milieu consisting of worksheets with text, commands, output and components (sliders, buttons, gauges ...). The slider trap in CAS-apps has a range of causes, some of which are:

- (1) The app reflects mathematics, which primarily is based non-digital understanding and standards.
- (2) The employed instrumented techniques have low epistemic value. The instrumental genesis requires little knowledge and methods resulting in superficial learning. The conceptual elements of the instrumented action scheme are not substantial with respect to the intended cognition.
- (3) The subject matter analysis (if any) behind the app is done independent of employed techniques.
- (4) There is no strategic planning of outsourcing to CAS.
- (5) Explorative features are not truly so. Outcomes of allowable manipulations are predefined.

### An illustrative example from calculus

A common CAS introduction of pointwise differentiation consists of a transcription of *the 3-step-method* into a computer environment, explicitly mentioned in official Danish guidelines: First form  $\Delta y = f(x_0 + \Delta x) - f(x_0)$ ; next reduce the difference fraction  $\frac{\Delta y}{\Delta x}$  to make the limit process  $\Delta x \rightarrow 0$  accessible. If the limit exists, then  $f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$ . If not, then  $f$  is not differentiable at  $x = x_0$ .

Mathematical software is not just computerized mathematics, but comes with educational intensions. The Maple app *Derivative Definition* is a response to teachers' needs, the professional solutions that spare them of cumbersome CAS coding. It consist of text and Maple components. The text comprises statements, descriptions and instructions, which slightly abbreviated read:  $\frac{df}{dx}$  at  $x$  is  $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$  if it exists, geometrically the slope of the tangent of the graph. One finds an approximation by ignoring the limit. The expression  $\frac{f(x+h)-f(x)}{h}$  is the slope of a secant. Use sliders and observe what happens,

when  $h$  approaches 0. The Maple components are a plot window displaying secants and tangents for a generic smooth function, sliders for the value of  $x$  and  $h$  and an output container showing the numerical value of the corresponding secant slope, see on-line version (Maplesoft, 2019). For this app we detail:

Re (1): The conception of the 3-step-method is predominantly algebraic paper & pencil methods underpinned by a few secant sketches. The epistemic potential lies in the work with the difference fraction and presupposes a (heuristic) limit concept. The app's refined plots do not give deeper understanding; they are just illustrations of a phenomenon. Students quickly realize this, but the app provides no further insight into why the phenomenon, originating from the 3-step-method, occurs.

Re (2): The app's epistemic potential for computation and understanding of point derivatives lies in approaching a certain line, called the tangent, by secants through sliding to zero. There is still a long way to go. The instrumented action scheme technically concerns to move sliders and conceptually to relate this to the geometric effect. The techniques give poor possibilities for output perception, for instance of properties of 'slope'.

Re (3): The core concept for the app is 'limit of secant slope', that is, instantaneous rate of change at  $x$ . It is represented in a semiotic register consisting of (English) language, mathematical symbols and non-instrumented transformations. Only after this is the setting converted to a new CAS-register with transformations from digital technology. Thus, the app mimics the teacher's non-instrumented perception, but is disconnected from the didactical rationale of the a priori analysis.

Re (4): Most of the work of the 3-step-method, algebraic manipulations of difference fractions, plotting and slope computations are outsourced to Maple components. Possibly this includes what may be considered as mathematical core activities. It requires fundamentally new scenarios to regain control, i.e. by returning to paper & pencil techniques. The Maple app itself is a hasty experience. In order that this is not by accident, a strategic planning of outsourcing is necessary.

Re (5): The app seemingly invites the students to explore the graphs by means of the Maple components, but there are no genuine opportunities for discovery. The app illustrates the definition, nothing more. In essence, the students might as well have watched a screencast.

### **The importance of a priori subject matter analysis**

The Maple app is an animation of the 3-step-method, which, in all likelihood, is more evocative than a few paper & pencil sketches. However, if teacher anticipated student learning is to get to substantial grips with existence and value of derivative, the trap falls. When one treats subject matter analysis and the instrumented techniques separately, the resulting milieu is meagre. In L's experience: "I have learnt a lot (analyzing the many steps in the three-step-method) about students' difficulties and will rework the project idea."

In a similar project targeting understanding of basic properties of differentiability (Grønbaek, 2016; G in the following), the core concept is again limit of secant slopes. However, along with the subject matter analysis G simultaneously considers computer possibilities for students' and designs a multiple faceted graph tool comprising several Maple component. The students use this tool in a series of exercises and problems composed for the purpose. From students' group discussions G concludes

that the concepts secant, tangent and  $f'(x)$  appear well understood, and that the visual and explorative approach has been a rewarding variety leading to discussions of core concepts.

A common outline of the a priori analysis for designing computer environments: Analyze on which core concept the target knowledge may build and subsequently decide on possibilities for computer treatment. The CMU-projects emphasize that the relation is mutual, one may have to change the core concept as well as the techniques. An example, also leading to knowledge about differentiation: If the core concept is local variation of function (rather than instantaneous rate of change), a computer environment naturally builds on zooming in on graphs. Zoom-in is the computer's version of  $\varepsilon - \delta$  argumentations and genuine tasks (i.e. without 'off-line interpretations') concerning approximation and limits are immediately at hand without the strict formalism: students can find the steepness of a graph in a point, ask and answer questions like: When is the zoom picture good enough? Can it be better? ... These possibilities relate to constraints of the computer such as resolution, pixilation, and runtime, pointing back to the need for an abstract formalism. The concept 'transposition informatique' of (Balacheff, 1993) captures the dialectic nature and educational implications of such transformations of the discipline of mathematics.

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# Calculus as a discursive bridge for Algebra, Geometry and Analysis: The case of tangent line

Irene Biza

University of East Anglia, UK; [i.biza@uea.ac.uk](mailto:i.biza@uea.ac.uk)

## Introduction

The tangent line to a curve is one of the mathematical topics that appear in different domains of mathematics very often with different uses. For example, we meet the tangent line in Geometry (e.g., tangent to a circle); in Algebra (e.g., tangent to parabola and other Cartesian curves); in Calculus (e.g., tangent to a function graph at a point where the function is differentiable) or in Analysis (e.g., tangent to a function graph as a line with the limit of the difference quotient as its slope). Research reports students' challenges with the tangent line to a function graph (e.g., Biza & Zachariades, 2010; Vinner, 1991). These challenges have been attributed, inter alia, to students' experiences with tangents in different domains of mathematics – for example, the tangent to a circle influences how students deal with tangents to function graphs.

In this short paper, I use the case of the tangent line to investigate the origins of students' difficulties when they learn topics in different mathematical domains including Calculus. To this aim, I draw on the commognitive framework proposed by Sfard (2008) that sees mathematics as a discourse and learning of mathematics as a communication act within this discourse. I start from the viewpoint that different mathematical domains endorse different discourses, namely they call for the use of different notation, are governed by different rules, and apply different definitions. Thus, if we see the tangent line as an object established in these different mathematical domains, most likely we speak about tangent line as a different discursive object in each domain. Very often, students are invited to learn and work with tangents while they engage with (and shift between) these discourses without being aware of these underpinning differences.

In my previous research on students' perspectives about tangent line (e.g., Biza & Zachariades, 2010), correct/incorrect characterisation of students' responses to tasks involving tangent lines led to a classification in groups with different perspectives (*analytical local*, *geometrical global* and *intermediate* between geometrical and analytical). In this paper, I return to the data from the same group of first year university mathematics students (Biza & Zachariades, 2010) in order to examine not only the correctness of the students' responses but also how they justify their choices and how different discourses are present in these justifications, regardless of their correctness or not – see a preliminary analysis in Biza (2017). I see Calculus as a crossroads between Geometry, Algebra and Analysis and argue that the lack of awareness of the differences in the transitions across these domains and their underpinning discourses can explain students' challenges with tangent line that research has reported repeatedly. I conclude the paper by highlighting potentialities of this analysis for teaching mathematical topics that are present in different mathematical domains, including Calculus, towards bridging the different mathematical discourses of these domains.



## **Tangent line from the commognitive perspective: An object in different mathematical discourses**

According to the commognitive framework (Sfard, 2008), mathematics is seen as a discourse which is established within a certain community. Mathematical discourse includes objects (e.g. the tangent line) and “discourses-about discourse” (p. 161), which are meta- rules about the use of these objects (e.g. what makes a line a tangent line). A mathematical discourse is defined by four characteristics: *word use*, *visual mediators*, *narratives* and *routines*. Word use includes the use of mathematical terms (e.g., tangent, derivative or direction coefficient) as well as everyday words with a specific meaning within mathematics (such as touch, region or point). Visual mediators are “visible object that are operated upon as part of the process of communication” (p. 133) and include mediators of mathematical meaning (e.g., function graphs, geometrical figures or symbols) as well as physical objects. Narratives include texts, written or spoken, which describe objects and processes as well as relationships among those (e.g., definitions, theorems or proofs), and are subject to endorsement, modification or rejection according to rules defined by a community (e.g., ‘a tangent line is a line that has one common point with a curve’ is an endorsed narrative for tangents in Euclidean Geometry but not in Analysis). Routines include regularly employed and well-defined practices that are used in distinct, characteristic ways by a community (such as defining, conjecturing, proving, estimating, generalising and abstracting). For example, identifying a tangent to a circle at a point A in Geometry means drawing a line, which is vertical to the radius at this point, whereas in Algebra this identification involves using the formula of the circle in the Cartesian plane and calculating the tangent line equation. In the commognitive frame, learning is seen as the development of discourse either at object-level (e.g., expansion of an existing discourse with new words and routines) or at meta-level, (e.g., changes in meta-rules). Very often, the teacher moves fluently between different narratives without communicating these differences explicitly (e.g., Park, 2015). Students, unaware of these differences, may be reluctant to change routines that worked well for them for new ones without seeing a reason for doing so. This reason is less transparent when teaching emphasises the *how* in the mathematical discourse, by mostly focusing on practical actions resulting in changing objects (e.g. how we calculate the formula of a tangent line), and with less attention on the *when* an existing or a new routine should be used. However, the *when* is exactly the aspect of an object and associated routines that can expand these routines in new ones or change them.

### **Methodology and Context**

Data reported in this paper were collected with a questionnaire administered to 182 first year university students (97 female) from two Greek mathematics departments. All participants had been taught about the tangent line in Euclidean Geometry (from Year 7), in Algebra (from Year 10) and in a Calculus with elements of Analysis course (in Year 12), but not yet at university as the study took place at the beginning of their first year. The questionnaire, inspired by previous works such as Vinner (1991), consisted of eight tasks in which the students were asked to: explain the tangent line in their own words (Q1); describe its properties (Q2); identify if a drawn line is a tangent line of a given curve (Q3); construct the tangent line, if it exists, of a given curve through a specific point (Q4 and Q5); provide the definition (Q6), write the formula (Q7), and apply the formula on specific functions (Q8) (Biza & Zachariades, 2008).



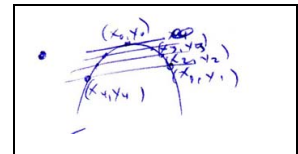
## Students' definition of the tangent line to the function graph

The analysis of students' justifications identified engagement with the different domains/discourses they had met tangents in: *Geometry*; *Algebra*; *Calculus*; and, *Analysis* as well as *Geometry-Local*, a *hybrid* discourse that endorses geometry narratives together with local meta-rules (e.g., “the line has one common point with the curve in a region of the tangency point”). Although this *hybridisation* was not in the curriculum, it did appear in student responses. Table 1, summarises these discourses with response examples. I note that students often engaged with more than one discourses in the same or across questions. For example: “ $A(x_0, f(x_0))$   $y - y_0 = \lambda(x - x_0)$   $f'(x_0) = \lambda$  direction coefficient. It is a unique line with only one common point near to  $A$ ” can be seen as both *Calculus* and *Geometry-Local*.

Discourse	Example (data have been translated from Greek)
Geometry	“No [it is not a tangent], the line has 2 points in common with the function graph”
Algebra	“Yes, the line $\varepsilon$ is tangent at $A$ , the slope equals to the direction coefficient [the coefficient $m$ in $y = mx + b$ , that indicates the slope of a line] of the line”
Calculus	“A function which is differentiable at a point $A(x_0, f(x_0))$ has a tangent at this point [...] its formula is $y - f(x_0) = f'(x_0)(x - x_0)$ ”
Analysis	“The line has formula $\varepsilon: y = \lambda x + \beta$ at the point $A(x_0, y_0)$ the point $A$ satisfies this formula [sic] and $\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lambda$ ”
Geometry-Local	“[It is a tangent, b]ecause if we consider a small region $(\kappa, \gamma)$ around the point $A$ where [the line] $\varepsilon$ is tangent we can see that [the line] $\varepsilon$ does not touch any other point”

**Table 1: Discourses identified in student responses with examples.**

For some students, working across discourses compromised the correctness of their responses. Other students, navigated across and within discourses with success. For example, S[149], a student who performed well in the questionnaire, writes in Q1: “[The tangent] is [the line] that has ‘two’ [his emphasis] common points with  $C_f$  the distance of which is infinitely small and thus we consider that it [ $C_f$ ] has a double point”. Then, in Q2, he adds: “ $f'(x_A) = \lambda$  the direction coefficient.  $f'(x_A) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_A)}{x - x_A}$ . At this point it



**Figure 1: S[149]’s response to Q2**

[the line] has one ‘double’ [his emphasis] common point with  $C_f$ . It can have other common points with  $C_f$ ,  $x \neq x_A$  [Figure 1]. In Q3.6, (inflection point) he writes: “The [line]  $\varepsilon$  is [tangent] because  $f$  is differentiable at  $x_A$  and  $\varepsilon$  has one (double) common point with  $C_f$  in the region  $(x_A - \kappa, x_A + \kappa)$ ,  $\kappa > 0$  and very small”. In Q3.7 (corner point where the function is not differentiable), he rejects the line because “ $C_f$  has two tangent semi-lines at  $A$  which, however, do not have the same slope”. We see in S[149]’s responses a mixture of *words* from Geometry (common points), Algebra (double point), Calculus (derivative) and Analysis (limits). A recurring endorsed narrative in his responses is the analytical definition of the tangent line through the limit of the difference quotient or/and the secants. All these words and narratives have been subsumed in the Analysis discourse, e.g., the *double point* is not seen as the algebraic solution but as the limiting position of two points that approach each other (“infinitely small”). As a result, the words are still used but this use is different – and is not contradictory. For S[149], drawing on the meta-rule of convergence (in Analysis) has shifted his meaning of common points, double point and derivative as discursive objects.

## Conclusions with teaching suggestions

The discursive analysis of students' responses indicated engagement with a range of discourses, from Geometry, Algebra, Calculus and Analysis and with a combination of discourses (see Table 1). Through this analysis, student responses that might at first have seemed incoherent (and very often incorrect), were explained, rationalised and demystified when seen in the context of student activities and experiences. Although previous studies on students' cognitive processes have created plausible explanations of students' thinking about tangents, a closer, commognitive look at students' justifications reveals potential origins of the challenges students face in the transition across mathematical domains. These challenges may originate, for example, in: the *applicability of routines* (Sfard, 2008, p. 215: a well-established routine may be evoked even if it is not appropriate) and *differences in discursive objects* (ibid, p.161: discursive objects may keep the same name but may have different uses and different meta-rules in different mathematical discourses).

Mapping out students' discursive activity through a commognitive lens suggests the potency of rethinking how we address students' difficulties, especially for topics students meet in different mathematical domains. First, considering the differences of these discourses is key in demystifying and addressing the challenges students often face. Second, conflicts between discourses is a significant part in students' learning and not a contingency that teachers may ignore or avoid. Third, not seeing mathematics as a homogeneous discourse and raising awareness of different discourses is essential in resolving such conflicts. Fourth, teaching with emphasis on mathematical definitions without discussing the rules on which these definitions are grounded may obstruct students from moving between discourses. Finally, engaging with a substantial range of examples in which a mathematical object is realised – in our case, tangency – is central to raising awareness of the different discourses in which this object is present. Appropriately selected examples act as catalysts between students' and teachers' discourses, can generate and resolve conflicts and offer a platform on which to discuss not only the *how* but also the *when* in mathematics, especially in cases such as Calculus which lies at a crossroads between Geometry, Algebra and Analysis.

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# Supporting the reinvention of slope

Rogier Bos, Michiel Doorman and Paul Drijvers

Freudenthal Institute, Utrecht University, Utrecht, the Netherlands; [r.d.bos@uu.nl](mailto:r.d.bos@uu.nl)

## Introduction

Introducing the slope of a curve in a point and the derivative of a function to students is a didactical challenge for teachers. It is tempting to choose for an instrumental approach, swiftly progressing from conceptual issues to differentiation techniques, both easier to teach and to learn. Without conceptual understanding this may entail merely meaningless manipulation of symbols and execution of recipes. The meanings of limit, difference quotient and other concepts related to the slope of a curve in a point form a serious obstacle for teaching this topic (e.g., Tall, 2013; Zandieh, 2000).

This paper discusses an attempt to deal with these obstacles inspired by design heuristics from the theory of Realistic Mathematics Education (RME) suggesting that a means to learn mathematics in a meaningful way is to engage students in a process of reinvention (Freudenthal, 1991). A common approach in RME is to introduce a new concept through a task in which the context provides opportunities for students to focus on reasoning and representations related to that concept. Gravemeijer and Doorman (1999) state it as follows:

The students should first experience a qualitative, global, introduction of a mathematical concept. This qualitative introduction then should create the need for a more formal description of the concepts involved. (p. 113)

This study reports initial experiences of an attempt to support students in (re)inventing the notion of the slope of a curve in a point and related representations (e.g. Zandieh, 2000). Students, collaborating in groups of three, are asked to design a playground slide (or a ski-jump) consisting of a bend and a straight part joining *without bumps*. The desired outcomes are concrete equations describing a line and a curve that meet smoothly (imagining the purpose being to feed these equations into a 3D-printer to print the slide). The task aims for students to discuss what it means for the line and curve to meet in a not-bumpy, i.e. smooth, way. They should search for methods to design such a slide and to decide to what extent smoothness is achieved. The hypothesis is that they will come up with ideas that are essential to the notion of slope of a curve in a point: informal building blocks, meaningful to the student, that can be exploited by the teacher to introduce the slope of a curve in a point more formally.

Students have a tactile (embodied) and visual understanding of what it means for a surface (or better, a curve) to be *smooth*, as also pointed out by Tall (2013). He suggests students slide their hands along a curve to sense the changing slope. In line with current views on embodiment, the slide task offers opportunities for students to mathematize this tactile and visual experience (Drijvers, submitted). A main merit of the slide task is that it is formulated in a very *open* way: it invites inquiry and various approaches. We discern three options: the standard textbook approach through secants of the curve, the linear approximation by a zooming in approach (Tall, 2013), and a more algebraic approach based on the multiplicity of intersection points. The most important design intention of this slide task is to

allow the teacher to connect the student's work to any or all of these approaches, depending on what students produce, possibly providing multiple views on the notion of slope.

The design of the lesson plan (scenario) for the task is based on the Theory of Didactical Situations (TDS) (Brousseau, 2002). We investigated whether combining RME and TDS as frameworks for the design of an inquiry-based mathematics lesson can lead to a successful scenario. This approach of combining ideas from RME and TDS to task design for inquiry-based mathematics teaching is explored in the Erasmus+ project Meria (Winsløw, 2017).

The research question addressed in this paper is: How do RME-inspired task characteristics and a TDS-inspired teaching scenario support students' reinvention of the notion of slope of a curve at a point? In this study the notion of *reinvention* refers to both the process of students' inventing solutions for the task and the whole teaching process of introducing the task and discussing the results by a teacher.

## Theoretical background

The design of the task originates from the RME-principle of a *didactical phenomenology* (Freudenthal, 1983). The challenge of realizing a smooth connection in the slide context is a phenomenon that begs to be organized by the tangent line; visually at first, but then also through symbolic means, to find out whether the candidate line and curve really join in a smooth way. This problem situation is expected to invite students to develop their own situation-specific solution methods. A (situational) model expected to emerge from the slide task is that of the slope of a curve at a point as the slope of a tangent line. This is a model *of*, produced after one episode of mathematizing. This is expected to develop into a model *for* further mathematizing towards symbolic and computational aspects of the slope in a point and the derivative (Doorman & Gravemeijer, 2009).

How to organize the classroom for such a reinvention activity? Freudenthal (1991) claims that "guiding means striking a delicate balance between the force of teaching and the freedom of learning" (p. 55). TDS might provide a suitable framework to balance the two. Central to the theory is the difference between didactical and adidactical situations (Brousseau, 2002). In a didactical situation the teacher acts intentionally to share his knowledge. In an adidactical situation the teacher purposefully withdraws, leaving space for students to develop their own activities. Students need this space to have the opportunity to invent their own meaningful strategies to address the slide problem. For Brousseau the alternation of didactical and adidactical situations also served the purpose of establishing a new *didactical contract*. During the adidactical situation the teacher cannot be expected to be involved in the mathematizing process. Instead (groups of) students interact with the *milieu*. For the slide task, the milieu consists of the problem itself together with the artifacts that may be needed to tackle it, for example GeoGebra.

The slide task lesson plan is set up according to phases from TDS and starts with the teacher explaining the problem and introducing artifacts that can be used to work on it. Then the teacher symbolically hands over the milieu to the students and withdraws. What follows is an adidactical *action phase* of 20 to 30 minutes. In this phase students work on the problem in groups of three. They may apply any approach they think is useful. The teacher, even though not interacting with the students, is not inactive: she registers solution strategies from the students and identifies examples

that might be used in the following *formulation phase*. The teacher makes sure that, for groups that have different strategies, one student explains their approach on the blackboard. Then follows a *validation phase*. The teacher asks questions such as: “are some solutions better than others? Is there a best solution? How do you know?”. The classroom discussion provoked by these questions form the input for the last phase: the *institutionalization phase*. In this phase the teacher is expected to be able to organize the ideas and strategies presented by students into solution models of the slide problem. The teacher makes a start with transforming the emerging *model of* the situation produced by students into a *model for* mathematical reasoning.

## Results from pilot studies

Pilot lessons have been conducted at schools in the Netherlands, Croatia and Slovenia with students who were not familiar with the derivative yet. Below we discuss indicators of *situational student models* observed in the action and validation phase, which have the potential to be developed into a more formal mathematical model in the institutionalization phase.

*Action phase.* In the action phase we observed several student strategies for the design. Obviously, students decided themselves on the type of curve they would use for the bent part of the slide. Then they needed to provide a way either to vary the line or the curve to investigate several designs. Some tried a discrete set of options, but others introduced parameters. Students working with GeoGebra dragged lines and curves, and some knew how to read the equations for the curves from the screen. Other students tried to draw a tangent line to the curve and created an equation from two observed points on the line. Occasionally students used two points on the curve (a secant line) resulting in an imperfect connection. There were two observations though (from Croatia) where students used this construction in GeoGebra and then moved one point towards the other. Students working with parameters were observed varying the parameters of the curve, e.g. the  $a$  in  $y = ax^2$ , or the parameters  $m$  and  $n$  of the line  $y = mx + n$ .

*Validation phase.* Students validated their designs both during the action phase and during the validation phase; in the latter they gave this more attention, encouraged by the teacher. Primarily most students evaluated the smoothness based on their intuitive, embodied idea of what that is. Some students talked about “a good fit”, some about tangent lines and some about intersection points. We observed three categories to classify students’ validation approaches:

- I. (Visual). Some relied on their visual evaluation of the design: if it looks good, then it is good. Students chose to zoom in on the curve (work on a smaller scale), for example using GeoGebra or similar.
- II. (Algebraic). Some computed whether their system of equations had the intended (unique) intersection point as a solution.
- III. (Numerical). Some designs were validated *by construction*, if numerical data obtained from drawings or GeoGebra were used to compute (the parameters for) their equations.

*Institutionalization.* Based on the informal student models the teacher had the opportunity to institutionalize various aspects of the notion of slope of a curve in a point. We sketch three cases:

1. (Intersection points approach) Many students focused on the fact that they want *no other intersection point near the intended intersection point*. This fits in nicely with a pre-Newtonian point of view to tangency and slope: the point of intersection corresponds to an algebraic solution with multiplicity  $\geq 2$  of equating the line and the curve. Using for example the discriminant the teacher could have shown a computation of this kind in a student design.
2. (Secant lines approach) In some cases students worked with two points on the slope, or their attention is drawn to the presence of another intersection point in an imperfect solution. The teacher then discussed how such a solution can be improved by moving the one intersection point towards the other. An improved solution was further discussed leading to a perfect solution (in a limit process).
3. (Linear approximation approach) Some students validated their design by zooming in. After zooming in long enough the curve will coincide with the line. These students seemed to be more focused on obtaining a “nice fit” and less specific on intersection points. A teacher then could have explained about local linearity of curves by using a simple example: suppose students have solutions  $y = x^2 + 2x$  and  $y = 2x$  intersecting in (0,0); without the higher order term  $x^2$ , the two equation indeed coincide. If the intersection point is not in the origin, this could be achieved by translation.

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# **Examining students' secondary-tertiary transition: Influences of history, disposition, and disciplinary engagement**

Kathleen M. Clark

Florida State University, USA; [kclark@fsu.edu](mailto:kclark@fsu.edu)

## **Introduction**

I have been interested in investigating undergraduate mathematics students' affective (emotional) experience with transitioning from school to university mathematics (e.g., the secondary-tertiary transition) for several years. I propose that assisting students in “confronting” the notion of transitioning from school to university mathematics has a role to play in students' perseverance in the major, and one way that this can be done is by using the historical development of particular domains in mathematics. Doing so enables students to engage with how the nature of mathematics has changed over time, and that this change is reflected in their own mathematical change (or transition) from the nature of mathematics at school (e.g., use of empirical methods) to the nature of mathematics at university (e.g., an abstract-formal orientation). The use of primary historical sources (in the form of Primary Source Projects, or PSPs) is one way in which students can meet these changes in the nature of mathematics – particularly when focusing on the different meta-discursive rules – and which has the potential to support students' meta-level learning.

## **Overview of the research plan**

The transition from school to university mathematics has received increased attention in recent years in a variety of contexts outside of the United States, which has been precipitated by mathematicians' decades-long lament about undergraduate students' lack of knowledge upon entry at university. Over time, this gap in knowledge has become known as the “transition problem.” Two aspects of the change from school to university mathematics are (a) the conceptual change and learning processes that students experience and (b) the transitions that take place as students move between social groups or contexts with different mathematical practices (Gueudet et al., 2016). In other words, *what* students learn and *how* they learn it changes dramatically between school and university mathematics, and these differences have a significant impact on students' decision to persevere as mathematics majors.

## **Interview component**

To examine the secondary-tertiary transition experienced by mathematics majors at Florida State University (FSU), I first seek to gain an understanding of features of the undergraduate mathematics major experience by conducting interviews of current mathematics majors (in the Pure Mathematics and Secondary Education tracks<sup>1</sup>). Of the 60 majors in our target population, 11 students responded to our request for interviews and I completed interviews for nine of these. The interview protocol was designed to glean as much information as possible, regardless of where students may identify themselves in the secondary-tertiary transition and each interview was approximately 75 minutes in length and video recorded. The interviews will capture student conceptions of their experience at

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<sup>1</sup> At FSU there are five mathematics “tracks” (or major programs): (Pure) Mathematics, Applied and Computational Mathematics, Biomathematics, Actuarial Science, and Mathematics/FSU-Teach (the secondary education major).

various timepoints<sup>2</sup> in the undergraduate mathematics major and will assist in identifying and describing attributes of the student experience at FSU. I completed the “exploratory interviews” in April 2019 and I will conduct the Transition Seminar in July 2019.

Although existing literature on the transition problem focuses on the causes for students’ difficulties in various transitions (Gueudet, 2008), there is evidence that identifying “relevant variables for a successful transition at the university level” (Di Martino & Gregorio, 2018, p. 4) is a fruitful enterprise to pursue. Among the variables Di Martino and Gregorio identified is self-concept as mathematics learners (and similar variables such as confidence and perceived competence), which typically holds a positive effect for students in their mathematical learning, and consequently continued perseverance in the mathematics major. Others have emphasized the importance of this positive influence (e.g., Rach & Heinze, 2017). Students often report increased levels of confidence when grades improve, or they similarly attribute poor grades to low levels of confidence with regard to mathematics. However, what has received less attention is determining relationships among confidence and perceived competence and other variables of interest, such as those proposed in this study.

### **Transition seminar component**

The second component of the pilot research study is to design, implement, and study a supportive intervention to address potentially critical aspects of transitioning to the study of undergraduate mathematics. Although this has been investigated a similar transition seminar (e.g., Witzke, Clark, Struve, & Stoffels, 2018), it was implemented in an institutional context quite different from FSU, and part of our inquiry is in fact to question whether the transition is “felt” or experienced in similar ways to that of other contexts, particularly in Europe. Furthermore, in the current pilot study I am interested in using the outcomes from the interview component to at least partially inform the design of the Transition Seminar – particularly with regard to what I learn from the interviews pertaining to affective dimensions, such as those related to perceived competence (or, confidence). The seminar will include the implementation of a sequence of PSPs which focus on (a) concepts from calculus that students would meet in a Calculus I course and (b) more abstract concepts that students would typically address in an Introduction to Analysis or Advanced Calculus course (at FSU), which is a senior-level mathematics course, but may be taken at any point after the completed prerequisites (e.g., Calculus III, Linear Algebra, and one “introduction to proof” course).

I intend to include within the Transition Seminar explicit discussions that address students’ emotional or affective experience within the transition, as well as to provide opportunities for students to “develop the more rigorous and critical view of the basic ideas of calculus that an introductory analysis course seeks to achieve” (Barnett, 2016, p. 294). Three different PSPs will be implemented during the seminar, which will provide opportunity to listen for how participants engage with the historical materials, including their ability to “develop an understanding of the language, techniques and theorems of elementary analysis that developed when mathematicians adopted such a critical perspective in the nineteenth century” (Barnett, p. 294) and can serve as an example of mimicking a

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<sup>2</sup> We targeted current mathematics majors who are at one of three timepoints along their trajectory of study: (1) taking courses in the Calculus sequence; (2) have completed the calculus sequence and at least one other prerequisite for Introduction to Analysis (or Advanced Calculus); (3) have completed Introduction to Analysis (or Advanced Calculus).



transition of sorts. For example, the ideas of analysis found in the calculus sequence at university (in the United States) can be perceived by students as a collection of procedures that are carried out for solving the problems of calculus (which may be more aligned with students' perceptions of school mathematics) – which is exactly what our interview findings support. However, in an introductory analysis course, the rules of the game appear to change, and students may feel they have fallen into an abyss of sorts, given the difference in the meta-level rules dictating this very different realm.

The pilot study is guided by several research questions which will enable a more precise description of the various features of the transition(s) that students experience, as well as how the emphasis on historical materials may mediate this experience:

- (1) In what ways are students' perceptions on the nature of calculus related to their affective and emotional response to transitioning from school to university mathematics, and what role do sources and materials from the history of calculus play with respect to students' perceptions, affect, and emotions?
- (2) How do seminar participants portray the differences between learning Calculus and Introduction to Analysis?
- (3) When using PSP materials, in what ways are students able to relate the transitions that occurred between 17<sup>th</sup> century and 19<sup>th</sup> century mathematics (e.g., from early calculus to analysis) with their own transition?

#### **A short note on PSPs**

The PSPs that are designed as part of the **TR**ansforming **I**nstruction in **U**ndergraduate **M**athematics via **P**rimarily **H**istorical **S**ources (TRIUMPHS) project employ a guided reading approach developed with support from the National Science Foundation (in the United States). In this approach, students are provided with “sufficient guidance to allow them to successfully read an original source, while still allowing them the excitement of directly engaging with the thinking of its author” (Barnett, 2016, p. 295). The curricular materials that will be used in the Transition Seminar will ask students to read selected excerpts in one PSP, for example, that:

motivate the need for a rigorous definition of the derivative, with some historical perspective. Newton's example of  $(x^m)'$  is also discussed by Cauchy later in the project. L'Hôpital's argument for the Product Rule with differentials is used to motivate a modern proof of the Product Rule for derivatives...near the end of the project. (Ruch, 2017, p. 10)

Another PSP includes:

writings of the nineteenth century mathematicians who led the initiative to raise the level of rigor in the field of analysis – as well as those who resisted or misunderstood this initiative – students' own understanding of and ability to work at the expected level of rigor can be refined. (Barnett, 2016, p. 295)

The PSPs will be implemented in an active-learning model, with the first PSP as a shorter (“mini-PSP”) designed to introduce students to the guided reading approach and to promote their interaction with other students in their discussion of concepts and results of their work on tasks. The second and

third PSPs will be similarly implemented and will focus on rigor in analysis. Student discussion and reactions will be elicited in order to address components of each of the three research questions.

### Acknowledgement

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# Introducing the integral concept with probability tasks

Charlotte Derouet

LISEC and ESPE of Strasbourg, University of Strasbourg, France;

[charlotte.derouet@espe.unistra.fr](mailto:charlotte.derouet@espe.unistra.fr)

## Introduction

In France, the teaching of integration begins the last year of high school, in grade 12 (17-18 year old students). We focus this paper on the teaching and the learning of the integral calculus and, especially, we present an original introduction of the concept of integral of continuous and positive functions in French high school, in the scientific track.

In the current French curriculum (in the scientific track), the integral of a continuous and positive function on an interval  $[a; b]$  is defined as the “area under the curve” (MEN, 2011). It is on this aspect of the integral that teachers must rely to introduce the new mathematical concept of integral. Our question is: how to introduce this concept in high school? Initially some textbooks propose to determine the area under a parabola. But why? It is an artificial problem: it is only a calculation of area to calculate an area... How motivating is the need to calculate an area under a curve really? With what problems? As Thompson and Silverman (2008) explain, the case where the integral is a measure of an area is the simplest case (where  $x$  and  $f(x)$  are lengths). The integral can be a measure of distance, of volume, of electric charge... In fact, the area under the curve is not the studied object but a register of semiotic representation (in the sense of Duval, 1993) of the quantity that it represents.

The original idea of this paper is to introduce the integral concept as an answer to probabilistic problems. Indeed, in the French curriculum of grade 12 in the scientific track, the register of semiotic representation of area under a curve is present in the part on continuous probability distributions too. We read that a random variable  $X$  (a function from  $\Omega$  to  $\mathbb{R}$  which associates for each result a number of an interval  $I$  de  $\mathbb{R}$ ) “fulfills the conditions that define the probability of the event  $X \in J$  as the area of the domain:  $\{M(x; y); x \in J \text{ et } 0 \leq y \leq f(x)\}$  where  $f$  is the density function of the distribution and  $J$  an interval included in  $I$ ” (MEN, 2011). In this special case, the area does not represent an area but a probability. There is the relation (\*):  $P(a \leq X \leq b) = \int_a^b f(t)dt$ . Our goal in this paper is to present designed tasks that create the need to calculate areas under a curve to determine probabilities and therefore to introduce a mathematical tool (in the sense of Douady, 1986) which addresses this problem with regard to the integral.

## How to understand the notion of accumulation through probabilistic problems?

Several researchers propose to use accumulation as the core idea for approaching the integral concept (Thompson & Silverman, 2008; Kouropatov and Dreyfus 2013, 2014). According to Thompson and Silverman (2008), “for students to see “area under a curve” as representing a quantity other than area, it is imperative that they conceive the quantities being accumulated as being created by accruing incremental bits that are formed multiplicatively” (p. 45). We think that aligns the area in a probabilistic context, to the accumulation approach.

In our proposition, three mathematical domains intervene to make sense at the integral: statistics, probability and calculus (Derouet & Parzysz, 2016) to pass from frequency to integral (table 1).

Statistics	$\Rightarrow$	Probability	$\Rightarrow$	Calculus
Descriptive statistics		Continuous probability		Integral calculus
Histogram		Probability density function		Continuous and positive function
Area of the rectangles		Area under the density curve		Area under the curve
Frequency		Probability		Integral

**Table 1: From frequency to integral**

In a histogram, we can determine the frequency  $f_i$  of a character on a class of amplitude  $a_i$  by the formula  $d_i \times a_i$  which  $d_i$  represents the frequency density on Y axis (see Derouet & Alory, 2018). Frequency corresponds to the area of the rectangles. The sum of the areas of all the rectangles is equal to 1. Probabilistic modelling of the phenomenon proposes to find a curve (of a probability density function  $f$ ) which “smooths” the histogram. Using an analogy between statistics and probability, we obtain that  $P(X \in [c; d])$  corresponds to the area under the curve on  $[c; d]$  and that  $P(X \in [c; c + \Delta x]) \approx f(c) \times \Delta x$  when  $\Delta x$  approaches 0. The bigger the considered interval, the higher is the probability (so the accumulation). The probability depends on the interval but also on the probability density (which depends to the interval). The integral is a tool to determine probability and after, the object “the integral of a continuous and positive function” is a generalization.

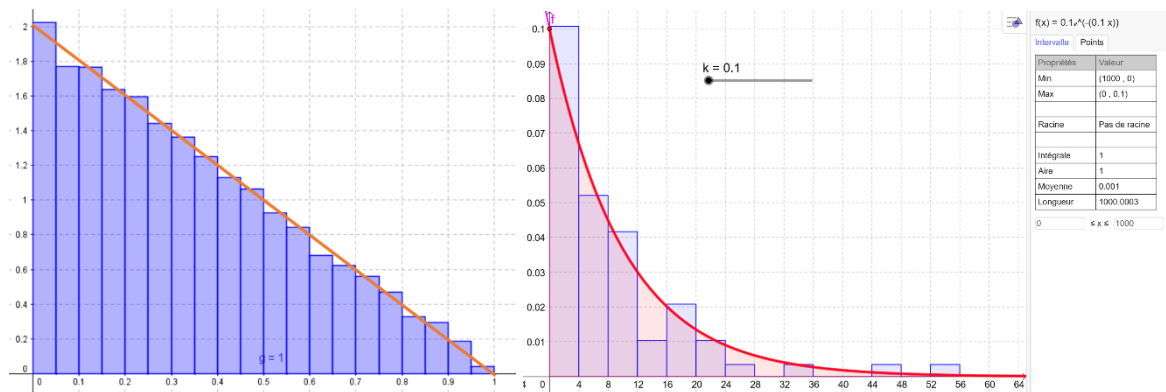
To enact progression in the introduction, we identify three levels of calculation of area under a curve:

- Level A1: calculation of elementary area. We determine the value of the measure of some basic geometrical objects (e.g. the area of a rectangle, a triangle...) that students know the formulas for, in other terms when the function is an affine function. In probability, uniform continuous distribution is an example of this level.
- Level A2: calculation of area with the use of the Fundamental Theorem of Calculus (FTC); in other terms when the function admits primitives that students knows (reference functions). In probability, the exponential distribution is an example. This level is the main goal of the chapter on integral calculus. Before the integration course, this level does not exist.
- Level A3: Approximate calculation of area using, for example, of the rectangle method or softwares. The normal distribution is an example of a distribution in this level (in grade 12).

## Designed tasks

We assume that students did not study antiderivative functions, integral calculus and continuous distributions before. The designed tasks are two modelling problems in probability. The first one considers the waiting time of the first people to arrive when two people have an appointment. The second problem is evaluating the time between two Aso volcano eruptions. The goals of these tasks lead to the notion of density function and its characteristics, especially the link between area under the desired curve and probability (and also that the total area under the curve is equal to 1) and the need to calculate different types of area under a curve. In the first problem, the area is of the level A1 (only elementary formulas of area formula are required) while in the second one, the density function

is negative exponential and so the level of the calculation of the area under the curve is A3 because students do not yet know integral calculus (figure 1).



**Figure 1: Histograms and density curves in the both tasks**

After getting the second curve, the problem then becomes: how to determine the area under this type of curve? The software GeoGebra is used to provide a first answer but students are then invited to find a solution to approximate the area in the interval  $[0; 20]$ .

## Implementation of the second task

A complete unit, articulating integral calculus and continuous probability distributions with these tasks, has been experimented on by different teachers since 2015 (Derouet & Alory, 2018). Derouet (2016) provides a detailed analysis of the first experimentation. We briefly explain the implementation of the second task. The students have a table with the years of the different eruptions of the Aso volcano between the 13<sup>th</sup> and 19<sup>th</sup> centuries and they must evaluate the probability that the next eruption will be within 5 years and during the year 2030 (the volcano erupted during the experimentation in March 2015). The first problem permitted to students to implement a step-by-step approach that enabled them to estimate a new time:

- 1) Gather the data in classes;
- 2) Represent them with a histogram (some students begin to draw it but the teacher then uses GeoGebra to construct histograms with different classes amplitudes);
- 3) Find a curve that “smoothen” the histogram (called a “tendency curve”) on the interval  $[0; +\infty[$ ;
- 4) Determine areas under the curve to evaluate probabilities.

In this problem, finding a curve that can be a candidate as a density function is not evident. Initially, the students proposed functions  $\rightarrow \frac{a}{bx+c}$ . They observed with GeoGebra that the area under the curve on the interval  $[0; +\infty[$  can not be equal to 1. Students wondered about the possibility or not that the area under the curve on an infinite domain is finite. They could refer to work done before, with examples of infinite broken lines, one of which has an infinite length and the other a finite length. Students then continued by proposing a decreasing exponential and they searched for appropriate parameters. GeoGebra is a numerical tool that permits the user to determine areas under a curve and so corresponding probabilities. The work then set out to approximate the area on the interval  $[0; 20]$ . Several techniques were identified in the class: counting tiles, compensation method, trapezium method, rectangle method, tangent method... Finally, the rectangles method is institutionalized (the

teacher coordinates work proposed by the students), and it is admitted that if  $f$  is a continuous and positive function on  $[a; b]$  with  $A$  the area under the curve than  $A$  is the limit of the Riemann sums. Then the integral is defined as the area under the curve. In a second step, the antiderivate function and the integral are linked by the FTC.

## Discussion

This approach aims to motivate the introduction of the integral of a continuous and positive function as a tool, before studying it as an object (Douady, 1986). To make sense to the area under a curve like a register of semiotic representation of an object (and not like the studied object), we propose an introduction with probability tasks using the idea of accumulation. Furthermore, we think that this introduction makes sense for density functions and permits one to connect integration and probability, that it is not the case in traditional teaching (Derouet, Planchon, Hausberger & Hochmuth, 2018) where continuous distribution is often only an application of integration formulas.

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# Real exponential in discreteness-density-completeness contexts

Viviane Durand-Guerrier<sup>1</sup>, Elizabeth Montoya Delgadillo<sup>2</sup> and Laurent Vivier<sup>3</sup>

<sup>1</sup>Université de Montpellier, France; [viviane.durand-guerrier@umontpellier.fr](mailto:viviane.durand-guerrier@umontpellier.fr)

<sup>2</sup>Pontificia Universidad Católica de Valparaíso, Chile; [elizabeth.montoya@pucv.cl](mailto:elizabeth.montoya@pucv.cl)

<sup>3</sup>Université Paris Diderot, France; [laurent.vivier@univ-paris-diderot.fr](mailto:laurent.vivier@univ-paris-diderot.fr)

## Introduction

In this proposal, we assume that working with undergraduate students on the triade discreteness – density – completeness is likely to foster conceptualisation in Analysis (Durand-Guerrier, 2016; Durand-Guerrier & Vivier, 2016). In her very complete review on the theme of discrete and continuum in secondary education in France, Rousse (2018) argues that the main problem at the secondary level is the understanding of density and that the completeness of  $\mathbf{R}$  is not an issue. It can then be hypothesized that while density is a major issue in the teaching and learning of calculus at secondary school, the distinction between density and completeness is an important issue of the transition from calculus to analysis at the beginning of university (Bergé, 2008).

A main difficulty with the conceptualisation of density as an intrinsic property of ordered number sets raised in Durand-Guerrier (2016) is to make visible in graphical representation the difference between the real numerical line (continuous line) and the rational (resp. decimal) one, the latter being both incomplete in the sense that there are, on the line, points without abscissa in  $\mathbf{Q}$  (resp.  $\mathbf{D}$ ), and not discrete because between two points with abscissa in  $\mathbf{Q}$  (resp.  $\mathbf{D}$ ), it is possible to put a different point with abscissa in  $\mathbf{Q}$  (resp.  $\mathbf{D}$ ). Due to the large use of graphical representations in calculus, this reinforces the common conception of a dichotomy between discreteness and completeness, that hinders the necessity of a completion of  $\mathbf{Q}$  (or  $\mathbf{D}$ ) for proving, without reference to geometrical intuition, main theorems of calculus (eg. The Intermediate Value Theorem, IVT).

We hypothesize that this is still reinforced by the use of digital tools, calculator, spreadsheet, grapher, that show signs, mainly numerical or graphical, that need to be interpreted for analytical work. The representation given by the artefacts is partial, truncated, and a reconstruction of the objects of analysis that are imperfectly represented is needed. There are two steps in this visualization: on the one hand, the fact of understanding that, despite the high precision of tools that makes it possible to obtain a significant number of decimals (at least 12 nowadays), there are potentially other non-zero decimals that we do not see; on the other hand, these tools cannot, despite appearances, justify the existence of an object that is based on mathematical properties. The first point is based on the notion of density of an ordered set, mainly in our study the set  $\mathbf{D}$  of finite decimal numbers, and the second mainly on the notion of completeness, of the set  $\mathbf{R}$  of real numbers. These two notions, which are mathematically very different, are treated in the same way by technological calculation tools with the production of the same signs.

We hypothesize that this could contribute to the important and various difficulties of teaching and learning analysis nowadays. Difficulties for students to acquire an understanding beyond the displayed signs and then to differentiate between notions. Knowledge involved is of two different and

intertwined types: mathematical knowledge and knowledge of the technological tools used. This led us to consider a didactic situation allowing students to visually distinguish between *density in itself* and *completeness*. Preliminary experiments conducted around exponential functions have led us to retain the definition of these functions based on their algebraic properties as a good candidate to a first reinterpretation of density with a removal of certain characteristics of the continuum, which should encourage the emergence of the notion of continuum.

We discuss first some results from a questionnaire showing difficulties of university students. Then we expose the didactic situation around the exponential function mentioned above.

### **Students' difficulties with the concepts of density and continuity**

The questionnaire was submitted at the second semester of the academic year 2015-2016 to 35 first year university students (group A1) and 69 students of second year university students (group A3) of the university of Montpellier. The results for group A1 are presented in (Durand-Guerrier & Vivier, 2016). There are completed here by the results for the group A3.

In the first question Q1, students were asked to provide, if possible, an interval with exactly two decimal numbers and to justify their answer. In both groups 20% did not answer this question, and 25% (resp. 13%) of students in Group A1 (resp. A3) provide an answer showing that density of **D** is not recognized (eg.  $[0.1;0.2]$ ). In the question Q2, students were asked to indicate what could be said of two numbers for which a calculator provided the same number with 12 digits. 63% of Group A1, and 88% of group A3 answered correctly that the distance between the two numbers was less than  $10^{-12}$ ; and 14% of group A1 and 6% of group A3 answered that the two numbers were equal.

It is noticeable that for both questions the results are better in second year, than in first year, while in general there is not a specific work on intrinsic density in the first year analysis course in Montpellier. An explanation could be that this is an effect of the high failure rate at the end of the first year, that means that those students facing strong difficulties do not access the second year. However, we cannot exclude the impact of using mathematics in other disciplines.

Question Q5 focused on the distinction between **D** (or **Q**) and **R**. Students were asked to decide whether there was a solution to equation  $f(x)=2$  in these sets of numbers, where  $f$  is a defined and continuous function on  $[-6.6]$  whose curve was given (the line  $y=2$ , not drawn, cuts once the curve of  $f$ ). This is a direct application of the IVT. Success stagnated with 74% for A1 and 68% for A3 with also no improvement in the confusion between density and completeness (affirmation of the existence of a solution in **D** or **Q**) with 23% for A1 and 26% for A3.

Question Q4, much more difficult than Q4, raised explicitly the distinction between density and completeness by discussing the existence of fixed points of a function from  $[0;1]$  to  $[0;1]$  given by a list of decimal values provided by a spreadsheet with two digits after the coma. In particular, the computed value for  $f(0.25)$  is 0.25. In the first case (Q4-a), the variation of  $f$  are indicated, but nothing is said about its continuity. As a consequence, the students cannot apply the IVT, so the answer is that it is not possible to decide whether there are fixed points or not. In the second case (Q4-b),  $f$  is assumed to be continuous. This allow to assert the existence of at least three fixed points in **R** because of the IVT. The rate of success for Q4-a is very low for both groups, and still lower for group A3



(23% for A1, 14% for A3). The rates are higher and similar for both groups for Q4-b (43% for A1, 42% for A3). There is a lack in taking into account the characteristics of the spreadsheet and the properties of the function: on the one hand, many students, probably interpreting the displayed value in the table for  $f(0.25)$ , answer that there is at least (or exactly) one fixed point (43% at Q4-a and 31% at Q4-b for A1 and 61% at Q4-a and 39% at Q4-b for A3); on the other hand, 31% of A1 students and 52% of A3 students give the same answer to both items.

Our results show that, in our population, a majority of students seem to have a correct understanding of the notion of density, but feel strong difficulties to grasp the distinction of this notion with the notion of the completeness. This supports our claim of the necessity of addressing specific work on this distinction. This is the main goal of the didactical situation we present below.

### **A situation to foster the distinction between density-in-itself and completeness.**

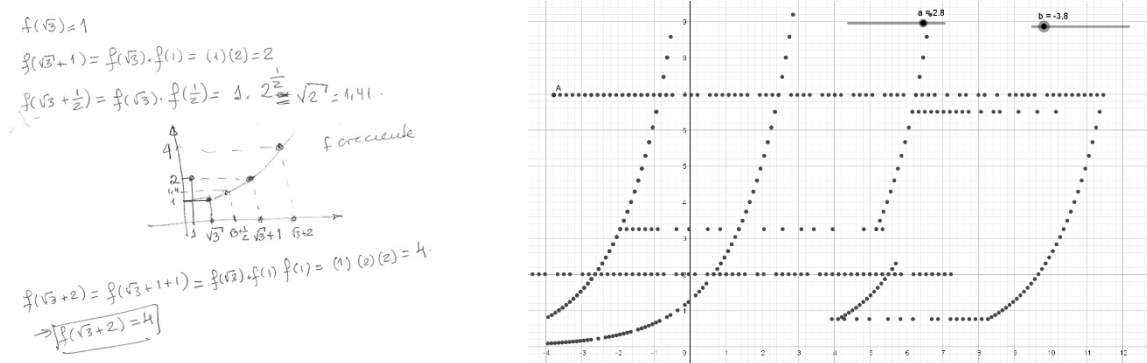
The mathematical question consists in searching for the functions  $f: \mathbf{E} \rightarrow \mathbf{R}$  satisfying the functional equation  $\forall x \in \mathbf{E} \forall y \in \mathbf{E} f(x+y) = f(x) \cdot f(y)$ , where  $\mathbf{E}$  is a usual numbers sets,  $\mathbf{Z}$ ,  $\mathbf{Q}$  and  $\mathbf{R}$ . We assume that  $f(1) = 2$  and it can be proved that  $f(0)=1$  and  $f$  takes positive values. We ask to compute some values of  $f$  (respectively, for each set, value of  $f$  at: 3 and -3;  $1/3$ ,  $-1/3$ , 1.3 and -1.3;  $\sqrt{3}$  and  $-\sqrt{3}$ ) and to draw graphical representations (on paper and with Geogebra software). A study in Chile with 6 voluntary students by pairs was conducted in 2017 and, with some changes for in-service teachers, in Peru in 2018. The data are made of written answers and direct observation in both contexts.

In the discrete case,  $\mathbf{Z}$  set, there is a unique function which is defined by  $f(n)=2^n$ . In addition to an induction, the functional equation must be algebraically used in order to use, for example,  $f(1+(-1))$  to find  $f(-1)=1/2$ . There is a relative ease to work, only algebraically, with the  $\mathbf{Z}$  set. Some graphical representations are drawn as a continuous line, but, with a commentary, quickly changed as a collection of discrete points. This phase is important in order to: (1) understand the problem; (2) produce a formula; (3) use the functional equation as a tool to compute values.

In the density case,  $\mathbf{Q}$  set, there are two difficulties. First, to compute values using algebraic properties of the functional equation such as  $f(a/b)^b = f(a)$  for two integers  $a$  and  $b$  ( $b$  non-zero). This technical difficulty can be skipped with helps, sometimes only on numerical values. Second, to draw the graphical representation for the  $\mathbf{Q}$  set. After the previous intervention on continuous lines in the discrete cases, the following question arises: do we draw a continuous line?

While, in the first sets, the function is entirely determined by algebra and the calculation of a value  $f(a)$  yields to  $2^a$ , this is no longer the case for  $\mathbf{R}$  where topological arguments, based on completeness, are needed and this constitutes a significant obstacle in the teaching of  $\mathbf{R}$ . The most advanced Chilean pair (3<sup>rd</sup> and 4<sup>th</sup> year), and some Peruvian teachers, tried to prove algebraically, which is impossible, that  $f(\sqrt{3})=2^{\sqrt{3}}$  and others assumed that for all  $x$  one has  $f(x)=2^x$  without a proof.

In Peru, after noting the failure to compute  $f(\sqrt{3})$ , it was asked to compute values of a function  $f$  that also satisfies  $f(\sqrt{3})=1$ . Some pairs computed values and put some points in a graph, linking them by a line (figure 1, left). The function  $f$  is not increasing as they wrote, but with their graph the problem can be stated. Indeed, they computed  $f(\sqrt{3}+a)$ , with  $a$  in  $\mathbf{Q}$ , and one has the same points than in the  $\mathbf{Q}$  case, but with a shift of  $\sqrt{3}$  to the right... still with the previous “ $\mathbf{Q}$ -points”.



**Figure 1: a graphic for for  $\mathbf{R}$ , Peruvian pair (to the left), a Geogebra graphic (to the right)**

Since 1 and  $\sqrt{3}$  are  $\mathbf{Q}$ -linear independent, hence: (1) one can assume any positive values at 1 and  $\sqrt{3}$  (and in any basis of the  $\mathbf{Q}$  vector space  $\mathbf{R}$ ); (2) the density of the additive subgroup of  $\mathbf{R}$   $\{a.1+b.\sqrt{3}; a \in \mathbf{Q}, b \in \mathbf{Q}\}$  yields that the graph of  $f$  is dense in the half plane  $y > 0$ . In Peru, it was asked by the searcher to investigate what happen in a Geogebra file with a point  $P(a.1+b.\sqrt{3}, 2^a)$  in drag mode. The result (figure 1, right) is like a chaos that made the teachers be surprised (it is not exp !).

## Conclusion

In this paper, we have tried to provide evidence that it is possible to work with university students on the distinction between density and completeness. Our study seems to show that there is a need of a specific work on this distinction for undergraduate students. This lead us to look for a didactical situation that helps in understanding the problem. We think that focusing more on calculations, formulas and favouring the use of a dynamic software would improve the situation.

However, to foster this distinction between these two properties, it is worth completing the situation. We hypothesize that looking for an increasing function satisfying the functional equation will enhance the completeness property (in that case, there is a single function which is the usual exponential  $2^x$ ). We have planned new experiments at various level of the university curriculum.

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# Teaching calculus with (informal) infinitesimals

Robert Ely

University of Idaho, USA; [ely@uidaho.edu](mailto:ely@uidaho.edu)

I describe a new approach to first-semester calculus using infinitesimals and I briefly summarize an assessment of its effectiveness in helping students interpret definite integral notation.

## Motivation and overview

For two centuries, Calculus was “the infinitesimal calculus.” For its inventors, G. W. Leibniz and Isaac Newton, it was a set of techniques for finding differential equations from regular variable equations and vice versa. In the 19<sup>th</sup> century the calculus was reformulated in terms of limits, so integration and differentiation could be done without having to postulate the existence of infinitely small quantities. 20<sup>th</sup> century calculus textbooks have almost exclusively followed this approach, using limits rather than infinitesimals, not because the latter were hard for students to learn but because they were seen as insufficiently rigorous (Thompson, 1914).

One result of this transition from infinitesimals to limits is that although calculus courses still use the notation of Leibniz, this no longer refers to the same underlying concepts he used it to refer to. For Leibniz,  $dx$  was an infinitesimal increment and  $\int$  was a sum (big S for “summa”) of infinitely many infinitesimal bits. In standard calculus classes nowadays, these notations are vestiges that no longer directly represent quantities. For instance, in  $\int_a^b f(x)dx$ , the big S does not directly denote a sum, nor is the differential  $dx$  a quantity, and derivative notation “ $dy/dx$ ” is code language for  $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$  rather than being a quotient of two quantities.

I believe that this shift in signification has contributed to some of the difficulties students encounter when interpreting and modeling with calculus notation. For example, students typically emerge from calculus courses with impoverished interpretations of definite integral notation. Most of the undergraduates interviewed by Jones, Lim, & Chandler interpreted the notation  $\int_a^b f(x)dx$  as either a gestalt area under a curve between  $x = a$  and  $b$ , or a call to find an anti-derivative for the function  $f(x)$  and evaluate it at  $a$  and  $b$ , or both (2016). Only 22% made any reference to summation of any kind, and less than 7% of them actually treated the notation as referring to a sum of pieces over a specified domain. Other studies have found similar phenomena (e.g., Fisher, Samuels, & Wangberg, 2016), despite evidence that sum-based interpretations of the definite integral are by far the most productive for supporting student modeling and interpreting (e.g., Sealey, 2006; Jones, 2015). But when  $f(x) \cdot dx$  is not really a bit of a quantity and  $\int$  does not really mean sum, is it surprising when students don’t readily interpret  $\int_a^b f(x)dx$  as a sum of bits?

The development of the hyperreal numbers in the 1960s makes it possible to ground infinitesimals in a secure rigorous mathematical foundation that was unavailable to Leibniz. A few calculus textbooks were written to this end (Keisler, 1986), but these are otherwise quite traditional and focused on “rigor” (formality). With this in mind, I have been piloting an *informal infinitesimals* approach to calculus. It uses infinitesimals and develops concept images for them like those used by Leibniz, with an eye to the formalizability of these later through the hyperreals (for more detail about

this formalizability, see the appendix of Ely 2017). One important concept image used for conceptualizing infinitesimals is an infinite microscope: zooming in infinitely on a point  $x$  reveals a little neighborhood or “monad” around  $x$  containing an entire world of new points that are all infinitely close to  $x$ . Zooming in infinitely again reveals a higher order of infinitesimals, each of which is infinitesimal even in comparison to ones at the previous scale. This image allows students to abstract from their intuitions about small quantities to develop a set of heuristics for rounding away higher-order infinitesimals, akin to those used by Leibniz, to derive differential equations and derivatives. I note that this image of zooming draws upon a recently-introduced kind of covariational reasoning: scaling-continuous covariation (Ely & Ellis, 2018). This contrasts with smooth-continuous covariational reasoning, which Thompson and Carlson (2017) describe as entailing an image of two quantities changing smoothly, a coordinated motion that is parametrized by underlying conceptual time. While smooth-continuous covariation models Newton’s reasoning with calculus, scaling-continuous covariation models the imagery Leibniz used. It uses the image that at any scale the continuum is still a continuum and never has holes or atoms, and that a variable takes on all values on the continuum. With this, one can imagine a re-scale to any arbitrarily small increment for  $x$ , coordinating that scaling with associated values for  $y$ . Ely & Ellis (2018) argue that scaling-continuous covariation can be as powerful for calculus students as smooth-continuous reasoning, but only in an approach that does not fundamentally rely on an image of motion.

This imagery allows differentials to directly represent infinitesimal bits of quantities, which lets students use their prior understandings to manipulate and interpret the notation. When  $dx$  is an infinitesimal increment,  $dy/dx$  really *is* a ratio of two infinitesimal quantities. This means the chain rule,  $\frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dx}$ , actually *is* cancellation of ratios. By invoking students’ prior knowledge of working with ratios, the notation can serve as a cognitive tool that suggests powerful and correct generalizations. This view is supported by recent work indicating that taking differentials seriously as quantities enables students to develop formulas through their own reasoning for volumes of rotation, work, and other ideas and applications in first-year calculus (Dray & Manogue, 2010).

## Two modes of interpreting definite integral notation using infinitesimals

In the informal infinitesimals approach, a definite integral really is a sum of infinitesimal bits. This approach explicitly provides students with two different modes for working with and interpreting definite integral notation: the adding-up-pieces (AUP) mode and the multiplicatively-based summation (MBS) mode. These ideas are adapted from Jones (2015), and are detailed in Ely (2017). An example can illustrate and motivate the distinction between the AUP and MBS modes:

By treating a curve as locally straight, we could see it as made of hypotenuses of right triangles with infinitesimal legs of lengths  $dx$  (uniform lengths) and  $dy$  (varying lengths). An arclength of such a curve from  $x = 0$  to 1 would be viewed as the sum of these hypotenuse lengths: (i)  $\int_{x=0}^1 \sqrt{dx^2 + dy^2}$ . But in order to *evaluate* this integral, by using the Fundamental Theorem of Calculus (FTC) and anti-differentiation (or even by direct computer estimation), the integral must first be converted to a different form: we imagine all the  $dx$ ’s as uniform in size and then being factored from the integrand, to get (ii)  $\int_{x=0}^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$ . If the curve is a function  $y = g(x)$ , the integral is now of the form

$\int_{x=0}^1 f(x) \, dx$  (where  $f(x)$  is  $\sqrt{1 + g'(x)^2}$ ). So the integral can be evaluated by  $F(1) - F(0)$ , for some  $F$  as an antiderivative of  $f$ .

Mode (i) is an example of adding-up-pieces (AUP): an integral  $\int_{x=a}^b dA$  represents the sum of infinitely many infinitesimal pieces  $dA$  over the domain  $x = a$  to  $b$  (in the example above,  $A$  is arclength). This mode is appropriate for modeling situations with definite integrals—one worries about evaluating the integral later. Mode (ii) is an example of multiplicatively-based summation (MBS): the integral has the form  $\int_{x=a}^b f(x) \, dx$ , so now there is an integrand  $f(x)$ , which is necessary for invoking the FTC. The integrand  $f(x)$  can be viewed as a rate at which  $A$  accumulates over the increment  $dx$ . If we are fortunate enough to recognize  $f(x) \cdot dx$  as an infinitesimal increment of any “amounts function”  $F(x)$ , we can find  $F(b) - F(a)$  to find out how much of  $A$  got accumulated between  $x = a$  and  $b$ . The MBS mode draws on Thompson’s (1994) development of the FTC by conceptualizing the integrand as a rate function, stressing the importance of students seeing the product  $f(x) \cdot dx$  as having the same dimensionality and quantitative type as  $A$ .

In a teaching experiment that used the informal infinitesimals approach, students demonstrated in interviews that they could effectively use the AUP mode on a modeling task with volumes, and that they could appropriately convert to the MBS mode (Ely, 2017).

### Assessing student interpreting of definite integral notation

I was curious how the informal infinitesimals approach would scale up to a large lecture, so in Fall 2017 I taught a (treatment) lecture for first-semester calculus, and a colleague taught a standard calculus course (control). My colleague is recognized as being a very effective and experienced calculus instructor. A pre-assessment and post-assessment were administered in both classes. The pre-assessment included only items that were possible to answer correctly without

	Item 3			Item 4			Item 8
	pre-	post-	gain	pre-	post-	gain	post-
Treatment (n = 92)	19.6%	69.6%	<b>+50.0%</b>	27.2%	65.2%	<b>+38.0%</b>	<b>91.3%</b>
Control (n = 133)	24.8%	51.9%	<b>+27.1%</b>	33.1%	37.6%	<b>+4.5%</b>	<b>58.6%</b>

**Table 1: Assessment score comparison on items pertaining to definite integrals**

<p><b>3.</b> The function <math>f(t)</math> provides the velocity of a moving car in miles per hour at time <math>t</math>. Suppose <math>\Delta t</math> represents a time increment of 0.2 hours. What does <math>f(t)\Delta t</math> mean in this situation?</p> <p>a) The total velocity of the car during a 0.2-hour period of time</p> <p>b) The integral of the velocity function for the car</p> <p>c) The change in the car’s velocity during a 0.2-hour period of time</p> <p>d) The change in the car’s position during a 0.2-hour period of time</p>	<p><b>8.</b> A truck is dumping sand onto a scale. At each time <math>t</math> (in seconds) the sand is dumping out at a rate of <math>f(t)</math> tons/sec. Which of the following best represents the total weight of sand, in tons, that dumps onto the scale in the first 4 seconds?</p> <p>a) <math>f(4)</math></p> <p>b) <math>f(4) - f(0)</math></p> <p>c) <math>\frac{f(4)-f(0)}{4}</math></p> <p>d) <math>f'(4)</math></p> <p>e) <math>\int_0^4 f(t)dt</math></p>
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**Figure 1: Sample assessment items**

having taken calculus—they used concepts of rate and accumulation that are focused on in calculus, but did not require calculus notation and techniques. Figure 1 shows sample items (Item 3 uses  $\Delta t$  instead of  $dt$  to be fair to the control class.) The post-assessment contained all these same items plus three more that explicitly used derivative and integral notation. Considering all the items on the pre-assessment, the pre-/post- gain in mean score was significantly larger in the treatment class (+34.78%) than in the control class (+15.64%) ( $\alpha = 0.05$ ). The scores for the particular items that pertain to definite integration are included in Table 1. These suggest that the students from the treatment class learned more than the control students about interpreting definite integrals.

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# Required knowledge of the derivative in economics: Results from a textbook analysis

Frank Feudel

Humboldt-University of Berlin, Germany; [feudel@math.hu-berlin.de](mailto:feudel@math.hu-berlin.de)

## Introduction and embedding of the research

Many students, who have to complete mathematics courses at university do not study mathematics as their major subject, but are enrolled in other disciplines. In these courses, the students need to learn the foundations of the mathematics required in their major discipline. This, in particular, includes knowledge of mathematical concepts that are important in these disciplines. However, literature shows that there are often discrepancies between the way mathematical concepts are taught in the students' mathematics courses and the way they are used in their major discipline (Alpers, 2017; González-Martin, 2018). These discrepancies might lead to difficulties when the non-mathematics students are trying to make sense of the mathematics used in their major subject (Christensen, 2008). Hence, it is important to investigate, which knowledge of mathematical concepts students in mathematical service courses need in their major discipline.

The research presented here focuses on economics, a discipline that had been rarely considered in mathematics education research. The mathematical concept chosen was the derivative because it plays a major role in economics, for example in economic theories like production or cost theory. The question 'what knowledge of the derivative do economics students need' was investigated by means of a textbook analysis because textbooks are often used as essential resources in economics courses.

## Methodology of the textbook analysis

Two important economics textbooks (in Germany) were analyzed: the book "Introduction to general business administration" by Wöhe and Döring (2013) and the book "Basics of microeconomics" by Varian (2011). Both are standard references for economics and cover material all economics students need to learn. The aim of this textbook analysis was to reconstruct the knowledge of the derivative economics students need for understanding the material presented in these books.

The chapters of the textbooks containing the derivative were analyzed with a two-step method based on a study by vom Hofe (1998), in which he investigated students' conceptions of limit with a task focusing on the transition from the difference quotient to the derivative. In a first step, the *description level*, vom Hofe described the students' statements related to the limit concept. In the following *explanation level* he tried to reconstruct the students' conceptions of limit by explaining their statements on the basis of limit conceptions described in literature.

For the textbook analysis presented here this two-step method by vom Hofe (1998) was adopted. In the first step, the *description level*, the information presented in each paragraph of the chapters containing the derivative was summarized. In the second step, the *explanation and reflection level*, this information was explained in mathematical terms, and analyzed with respect to the question:

"What knowledge of the derivative concept is needed to understand the material presented?"



## Some Results of the Analysis

Due to limited space, I will restrict the presentation of the results to three examples from the textbook by Wöhe and Döring (2013) discussed in the following subsections.

### The use of the derivative as “marginal function”

The derivative occurs in the book by Wöhe and Döring (2013) for the first time in the chapter about cost (p. 293-305), in which it is introduced as marginal cost. The content concerning the use of the derivative as marginal cost is typical for its use as “marginal function” of other economic quantities like revenue, profit or utility as well.

*Description level:* The chapter starts with the introduction of cost functions:  $C(x)$  represents the cost if an output  $x$  is produced. Thereby, different notions of cost are introduced (p. 299-303). Relevant with respect to the derivative is the *marginal cost*, which is defined as follows (p. 300):

The marginal cost  $C'$  is the cost of the last unit. Hence, you can determine the marginal cost of the 33th unit by subtracting the total cost of 32 unit from the total cost of 33 units. The marginal cost represents the slope of the original cost function. It can be determined by taking the derivative of the original cost function:  $C' = \frac{dC}{dx}$ .

*Explanation and reflection level:* What is going on here is that the derivative is identified with its economic interpretation (here as cost of the last unit). In order to make sense of this definition of marginal cost, which may be contradictory if the students just relied on their knowledge concerning the derivative from school, the students need to know that the derivative is used as a linear approximation of the cost of the last unit here. Furthermore, the students need to know that in this identification *one unit* is assumed to be so small in the considered context that the error between  $C'(x)$  and  $C(x) - C(x-1)$  is negligible (Wiese, 1999).

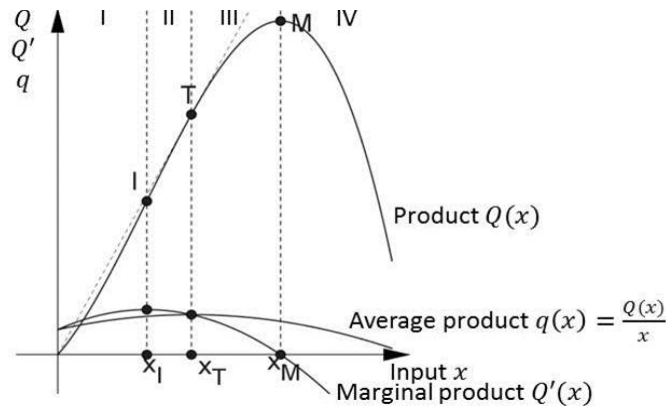
### The use of the derivative for the description and characterization of economic functions

The next chapter in the book containing the derivative is the chapter about production and cost functions (p. 305-309). Here, the derivative is used to describe and characterize so-called “classical production and cost functions”. The use of the derivative in this chapter is typical for its use to describe other economic functions in the economic theory as well.

*Description level:* Wöhe and Döring (2013) first introduce the “classical production function” in the context of the law of diminishing returns in agriculture (p. 305). If one increases the production factor “labor” by holding other factors, like the seeding material or the fertilizer, constant, the output first increases with increasing marginal product, in a second phase it increases with a decreasing marginal product, and finally decreases itself. Hence, the “classical production function” that follows this law has different stages. Wöhe and Döring illustrate them in a diagram like Fig. 1.

They then describe the relationship between the product function  $Q$ , the marginal product  $Q'$  and the average product  $q$  with  $q(x) = \frac{Q(x)}{x}$  in the different stages. For example: In the second phase after the inflection point, the function  $Q$  is increasing and concave,  $Q'$  decreases, and the average product  $q$  increases until it reaches its maximum at the tangential point  $T$ , in which the ray from the origin equals the tangent of  $Q$ . Explanations why these relationships are valid are not given.





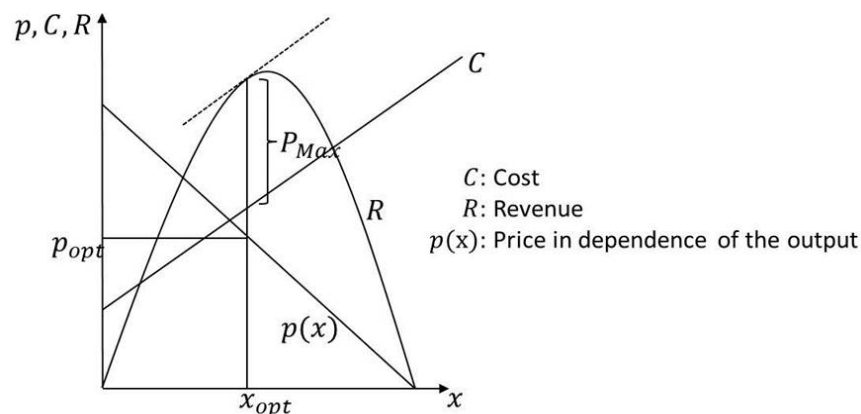
**Fig. 7: Stages of the classical production function like in Wöhe and Döring (p. 306)**

*Explanation and reflection level:* To understand the relationship between  $Q$  and  $Q'$  in Fig. 1 the students need to know the connection between the derivative and monotonicity/convexity. To understand why the graphs of  $Q'$  and  $q$  intersect in the tangential point  $T$  (reason:  $q(x)$  is the slope of the ray from the origin through  $(x, Q(x))$ , which equals the tangent at  $T$ ), students need to know the geometric representation of the derivative as slope of the tangent line, and they must be able to determine the slope of a linear function. Finally, to justify why  $q$  reaches its maximum at  $x_T$ , one needs the differentiation rules and the condition  $f'(x) = 0$  for optimal values because the function  $q$  has a stationary point just at  $x_T$  ( $q'(x_T) = \frac{Q'(x_T) \cdot x_T - Q(x_T)}{x_T^2} = \frac{Q'(x_T) - q(x_T)}{x_T} = 0$ ).

### The use of the derivative for optimization in price theory

The aim of the price theory is to determine the price and the output for maximal profit of a company. The profit can be calculated as difference of revenue and cost:  $P(x) = R(x) - C(x)$ . The revenue is determined as product of price and output  $p(x) \cdot x$ , whereas the price that can be set also depends on the output that is produced ( $p(x)$  is, for example, assumed to be strictly decreasing in a monopoly market).

*Description level:* Two methods are illustrated in Wöhe & Döring (2013) to determine the optimal output and the optimal price (p. 422-423). The first one is the use of a table of values, which does not rely on the derivative. The second method is a graphic one. It is illustrated in Fig. 2. No explanation is given, why the optimal output, price and the maximal profit are determined this way.



**Fig. 2: Graphical method in Wöhe and Döring (2013) for finding the optimal output and price**

*Explanation and reflection level:* The graphical method of Fig. 2 is to apply a shift on the cost function until becomes the tangent line to the revenue function. The method works because at the point  $x_{opt}$  with maximal profit the derivative of  $P'$  equals to zero. Therefore  $R'(x_{opt}) = C'(x_{opt})$  and the slope of the tangent line of  $R$  equals the slope of  $C$ . To understand why this graphical method works, the students therefore need to know the geometric representation of the derivative as slope of the tangent line and the necessary condition  $f'(x) = 0$  for optimal values.

## Summary and discussion

The examples presented here already show the necessity of the following knowledge concerning the derivative: 1) the economic interpretation of the derivative and its connection to the mathematical concept via linear approximation, 2) the geometric representation of the derivative as slope of the tangent line, 3) the connection between the derivative and monotonicity/convexity, 4) criteria for optimal values involving the derivative, and 5) the differentiation rules. In addition, the textbook analysis showed that the students also need to know the definition of the derivative (which could not be illustrated here due to limited space). Hence, economics students need a lot of knowledge concerning the derivative beyond calculation procedures. This knowledge was then needed to find answers or understand presented solutions to qualitative, theoretical problems.

The way the derivative is used in economics textbooks, however, often does not coincide with the way it is taught in the mathematics textbooks. The latter mainly focus on calculation procedures. To reduce students' difficulties when trying to make sense of the way the derivative is used in economics, the mathematics teaching should not just focus on these procedures to solve concrete calculation problems but also provide a rich knowledge of the derivative as a mathematical concept, that can be used to answer qualitative, theoretical problems like the ones presented here.

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# What happens after Calculus? Examples of the use of integrals in engineering: the case of Electromagnetism

Alejandro S. González-Martín<sup>1</sup> and Gisela Hernandes-Gomes<sup>2</sup>

Département de Didactique, Université de Montréal, Canada;

<sup>1</sup> [a.gonzalez-martin@umontreal.ca](mailto:a.gonzalez-martin@umontreal.ca); <sup>2</sup> [gisela.hernandes.gomes@umontreal.ca](mailto:gisela.hernandes.gomes@umontreal.ca)

## Introduction

Research on mathematics as a service course has been attracting more and more attention in recent years. This may be due in part to the high enrolment numbers seen in graduate mathematics courses (calculus in particular) driven by the lure of a STEM career. It could also be due to the high failure and dropout rates in these courses and their associated programs (Artigue, Batanero, & Kent, 2007; Faulkner, Earl, & Herman, 2019; Rasmussen & Ellis, 2013). Research has identified a gap between the content presented in mathematics courses and the way this content is used in professional courses (Christensen, 2008). This can leave students – and stakeholders – questioning the pertinence of mathematics course content.

Currently there is scant research examining how mathematical content is used in professional courses. We aim to close that gap. Our recent work focused on how integrals are used in a civil engineering course (Strength of Materials) to define bending moments for beams (González-Martín & Hernandes Gomes, 2017) and to define first moments of an area and centroids (González-Martín & Hernandes Gomes, 2018). Our results show that although integrals are used to define these notions, the methods used to solve tasks do not require techniques taught in prerequisite calculus courses. Rather, the conceptual aspects of integrals are used to introduce and define these notions (in particular, their interpretation in terms of area), and the assigned tasks mostly employ geometric calculations and ready-to-use formulae. We aim to expand our analysis to other engineering courses in which integrals are used to define notions, to determine whether this is a common phenomenon. In particular, we seek to identify how much calculus content is required to introduce these notions, and whether the tasks employed require the techniques learned in calculus courses. In this paper, we discuss our ongoing results concerning the content related to electromagnetism in an electrical engineering course, General and Experimental Physics.

## Theoretical framework

We consider professional and mathematics courses in engineering to be different institutions, each with its own set of tasks and practices. In analysing practices, we employ tools from the anthropological theory of the didactic (ATD – Chevallard, 1999), specifically the notion of praxeology. A praxeology  $[T/\tau/\theta/\Theta]$  has four elements: a task  $T$  to perform, a technique  $\tau$  which allows the task to be completed, a rationale (technology)  $\theta$  that explains and justifies the technique, and a theory  $\Theta$  that includes the discourse. To analyse what an institution considers as ‘knowing the content  $X$ ,’ we need to analyse the tasks that use  $X$ , as well as the techniques and rationales (both implicit and explicit) used to justify these techniques. ATD distinguishes between different types of praxeology, but due to space constraints we are limiting our presentation to tasks and techniques.

It should be noted that ATD recognizes that knowledge and praxeologies can be created in one institution but used in another. This process imposes a *transposition* effect on the praxeologies in question (Chevallard, 1999), in which some (or all) elements of the original praxeology change and evolve. We consider this construct to be especially useful, particularly because we are interested in identifying possible gaps between the content of calculus courses and its use in engineering courses.

## Methodology

We are currently analysing reference books used in various engineering courses. In this paper, we consider Halliday, Resnick, & Walker (2014), used in General and Experimental Physics, and Stewart (2012), used as a Calculus textbook. We proceeded in two stages: First, we analysed the general structure of the content related to integrals in Stewart (2012), identifying the main tasks concerning integrals as well as the techniques and rationales (technology) used to solve these tasks. Second, we analysed Halliday et al. (2014), identifying the notions defined as an integral and the tasks that involve these notions, as well as the techniques and explanations (technology) presented. This allowed us to pinpoint the different praxeologies where integrals are used in both courses.

## Data analysis

To date, we have identified several notions that are defined as an integral in the engineering textbook (Halliday et al. (2014), including the following: electric field ( $E$ ), electric flux ( $\Phi$ ), electric potential ( $V$ ), Gauss' law, potential energy ( $U$ ), and work ( $W$ ).

In the Calculus textbook, the content regarding integrals is organised into two main blocks (González-Martín & Hernandez-Gomes, 2017, 2018): the first block introduces techniques for calculating indefinite integrals (immediate integration to begin with, followed by various integration techniques). We note that theoretical elements justifying the different integration methods are mostly absent. The second block introduces Riemann sums to formally define integrals and interpret them as areas, and leads to the Fundamental Theorem of Calculus and the calculation of definite integrals using Barrow's rule; the book then provides applications of the integral (area, volume, etc.). Many of the techniques used in this second block are based on elements explored in the first one.

Regarding the General and Experimental Physics textbook, to date we have analysed the chapters on electromagnetism (chapters 21 to 24). Our results are similar to those of our previous research concerning a Strength of Materials textbook (González-Martín & Hernandez-Gomes, 2017, 2018). Integrals are used mostly to introduce and define notions proper to electrical engineering, and it is rather the interpretation of an integral that allows for a proper analysis of the phenomena studied. For instance, the definition of the electric potential  $V$  at a point  $P$  in terms of the work  $W$  done by the electric force and resulting potential energy  $U$  is  $V = -\frac{W_{\infty}}{q_0} = \frac{U}{q_0}$  (Halliday et al., 2014, p. 686).

Moreover, the work done by a general variable force can be calculated by a definite integral:  $W = \int_{x_i}^{x_f} \mathbf{F}(\mathbf{x}) d\mathbf{x}$  (p. 163). This means that both work ( $W$ ) and electric potential ( $V$ ) are defined using integrals. Regarding the tasks, most techniques call for the use of given properties or tables, which means that students can solve them without consciously using integrals (as in the Strength of Materials course). For instance, Figure 1 shows one solved exercise that uses ready-to-use formulae.

The technique required to solve this task makes use of an available equation (24-41:  $E_z = -\frac{\partial V}{\partial z}$ , p. 702), and the derivation of a square root and a linear term.

**Sample Problem 24.05 Finding the field from the potential**

The electric potential at any point on the central axis of a uniformly charged disk is given by Eq. 24-37,

$$V = \frac{\sigma}{2\epsilon_0} (\sqrt{z^2 + R^2} - z).$$

Starting with this expression, derive an expression for the electric field at any point on the axis of the disk.

**KEY IDEAS**

We want the electric field  $\vec{E}$  as a function of distance  $z$  along the axis of the disk. For any value of  $z$ , the direction of  $\vec{E}$  must be along that axis because the disk has circular symmetry

about that axis. Thus, we want the component  $E_z$  of  $\vec{E}$  in the direction of  $z$ . This component is the negative of the rate at which the electric potential changes with distance  $z$ .

**Calculation:** Thus, from the last of Eqs. 24-41, we can write

$$E_z = -\frac{\partial V}{\partial z} = -\frac{\sigma}{2\epsilon_0} \frac{d}{dz} (\sqrt{z^2 + R^2} - z)$$

$$= \frac{\sigma}{2\epsilon_0} \left( 1 - \frac{z}{\sqrt{z^2 + R^2}} \right). \quad (\text{Answer})$$

This is the same expression that we derived in Module 22-5 by integration, using Coulomb's law.

**Figure 1 – Example of problem involving electric potential (Halliday et al., 2014, p. 702)**

However, we note that the expression for  $V$  is deduced in a theoretical section using integrals:  $V = \int dV = \frac{\sigma}{2\epsilon_0} \int_0^R \frac{R' dR'}{\sqrt{z^2 + R'^2}} = \frac{\sigma}{2\epsilon_0} (\sqrt{z^2 + R^2} - z)$  (p. 700). This integral is calculated employing a technique calling for the use of a table of antiderivatives provided at the end of the book. We note that Stewart (2012) calculates this type of integral using trigonometric substitution (p. 478). Therefore, although the section on Electromagnetism in Halliday et al. (2014) uses more complex functions than the Strength of Materials book, they only appear in the theoretical sections, their antiderivative is calculated using tables found at the end of the book, and only the result is needed to solve problems.

## Final remarks

Our previous results regarding the notions of bending moment and first moment of an area (González-Martín & Hernandez-Gomes, 2017, 2018, respectively) indicated that these two notions, although defined using integrals, are used in praxeologies that are quite different from typical calculus course praxeologies. Curious about whether this phenomenon is repeated in other engineering courses and in other topics, we analysed the reference book used in another course, in a different subfield of engineering.

While still incomplete, our results point to the same conclusion: the praxeologies of this course are very different from praxeologies found in calculus courses, which may be at the origin of the gap identified by researchers (Christensen, 2008). Our partial results indicate that although the Strength of Materials and General and Experimental Physics courses use integrals to define notions proper to engineering, the justifications given do not go into the same level of detail as in calculus courses. Indeed, some properties are simply stated as a rule, without any explicit justification (which could be provided using tools from calculus). Regarding the tasks to solve, the books' techniques usually call for the use of geometric considerations, tables and ready-to-use formulae. As for the small number of tasks that require an integral to be computed, they either call for very simple functions or provide tables for students to calculate the antiderivative.

Our results seem to confirm the gap identified in the literature. We believe that by analysing the content of professional courses and understanding how calculus content is used in them, we can better

address questions concerning the adequacy of calculus courses in the training of engineers. Given the failure and dropout rates in engineering programs (and in STEM programs in general), research on these issues is urgent.

### Acknowledgments

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## A vision for the future of college calculus

Jessica Ellis Hagman

Colorado State University, United States of America; [jess.ellis@colostate.edu](mailto:jess.ellis@colostate.edu)

For the past decade, I have been a part of a large research team studying college calculus. This research team has been led by David Bressoud, run under the auspices of the MAA, and funded by the NSF. Our research has come from two projects, the first (*Characteristics of Successful Programs in College Calculus (CSPCC)*) begun in 2009 and focused on mainstream college differential calculus programs (typically called Calculus I) at all institution types; the second (*Progress through Calculus (PtC)*) begun in 2014 and focused on precalculus, differential and integral calculus programs at Masters and PhD-granting institutions. Our work has been generally focused on identifying aspects of college calculus programs that are successful or innovative, and supporting more mathematics departments as they work to improve their programs based on these findings. Overall, we did not see great evidence of success in college calculus across the US. Among the students surveyed, we saw significant decreases in confidence, enjoyment, and interest in continuing to study mathematics (Bressoud et al., 2013), and we found that nearly 18% switched out of the calculus sequence after taking differential calculus, with women switching significantly more often than men of similar preparation and experiences (Ellis, Fosdick, & Rasmussen, 2016).

From the 213 schools that participated in the CSPCC survey, we identified 18 schools that showed promise, including community colleges, Bachelor's-granting, Master's-granting, and PhD-granting schools. We conducted case studies at these sites and, based on these case-studies, have identified a number of components of calculus programs potentially related to student success. From the five doctoral-granting departments we visited, we identified seven features that were common and that we believed were related to their program's success and that worked together to engender student success (Rasmussen, Ellis, Zazkis, & Bressoud, 2014). These seven features include (1) a coordinated program where (2) instructors, especially novice instructors, are supported to teach and to develop a sense of shared ownership over the course; (3) the students are placed into the highest course in which they can succeed, (4) the curriculum actively engages students in (5) cognitively demanding tasks, and (6) there is support (such as calculus specific tutoring centers) towards students' success in these courses; and (7) the department collects data about multiple aspects of the program to understand areas for improvement.

Since publishing those findings, we have seen a number of calculus programs across the US use these findings to guide improvements in their own programs, showing the impact that such studies can have on shifting the national landscape of calculus education. However, I have recently argued that by focusing on these seven characteristics alone, departments may foster inequities by further supporting the populations of students who are already successful in calculus. Instead, I argue that departments should explicitly attend to diversity, equity, and inclusion while also improving their programs through focus on the seven characteristics (Hagman, accepted).



## **The Future of College Calculus: Respond to today's students' needs**

Through the PtC work, I hoped to find a mathematics department where the calculus program was thoughtfully crafted to best support today's college students – a more diverse population of students, that includes more students of color, and more first-generation and low-income students than before (Eagan et al., 2014). We did not find a program that had an explicit focus on supporting a diverse population of students to thrive in mathematics, but we did see a calculus program developed to support every student in their construction of mathematical meanings in calculus. This program was developed based on research rooted in radical constructivism, and not with an explicit attention to equity. However, I believe this program affords an anti-deficit approach to mathematics by viewing every student's mathematical understanding as valuable and part of the construction of richer mathematical meanings. This calculus program illustrates that by sincerely valuing every student's mathematical understandings, and leveraging research to support each student's rich construction of mathematical meaning, a diverse population of college calculus students can mathematically thrive.

### **Background on DIRACC**

Project DIRACC (*Developing and Investigating a Rigorous Approach to Conceptual Calculus*) is an NSF-funded college calculus curriculum developed by Pat Thompson and his colleagues based on years of research on student understandings of calculus (see Thompson, Byerley, and Hatfield (2013) for a description). This curriculum is self-described as “Newton meets Technology”, focusing on developing meaning for infinitesimals and differentials (while utilizing animations and interactive apps) rather than emphasizing the notation and formality of Leibniz. This curriculum is shared online for free, and is currently being implemented in at least two large, public, doctoral granting mathematics departments, including one involved in Progress through Calculus. In this short report, I will draw on my experience at the one university involved in PtC (referred to as Large State University; LSU), where DIRACC is the curriculum used for all calculus courses for science, computer science, and mathematics majors. The undergraduate population of LSU is approximately 50% white students, 20% Hispanic and Latinx students, 7% Asian, and 5% Black and African American. In the DIRACC calculus courses I observed, I estimated that approximately 30% of students were Black, Latinx, and/or Native (based on appearance). At LSU, there is a separate (and more procedurally oriented) college calculus course for engineering majors. The DIRACC courses are taught by instructors, mathematics education faculty, and doctoral students pursuing degrees in pure and applied mathematics and mathematics education. This course is coordinated by a full-time instructor, and this coordination includes weekly meetings for all instructors, where the topics of discussion during the meetings include understanding the mathematics and student thinking related to the mathematics for the upcoming section. Preliminary results from PtC indicate that students in DIRACC outperform students at comparable universities on a calculus content assessment and maintain positive beliefs towards mathematics more than students at other institutions.

### **Shift in Curriculum**

To best serve the students in our calculus classes, we need to learn what is motivating them to pursue degrees requiring calculus – whether future career goals or general interest in learning – and rethink our calculus curriculum to be in line with these interests. It is well established that in today's



economy, STEM jobs pay significantly more, on average, than non-STEM jobs (NSB, 2018). Given this widespread knowledge, we cannot ignore that one contributing motivation for students to pursue STEM is future job and wage prospects. When sitting in Calculus I classes across the country, it often seems that everyone knows the students are there not to learn deep and interesting mathematics, but to get a grade in the course that allows them to continue pursuing whatever STEM degree they are hoping for in order to get a good job. I believe that we are missing a big opportunity in our calculus classes to inspire these STEM-intending students about the magic and beauty of calculus. The great majority of calculus courses I have visited have been “mainstream” courses, meaning to serve all STEM students, although in actuality the great majority of the content is driven by the needs of the engineering students, with occasional word problems being set in other contexts.

In a forward-thinking calculus system, there would be a meaningful connection between the content taught in calculus, the needs of the majors whose students are taking calculus, and the interests and motivations of the students enrolled in our courses. It would be these latter two driving the content, rather than historical precedents. The DIRACC curriculum achieves this by forgoing Leibniz’s precise notation in favor of Newton’s more intuitive ideas – skipping the formalities of ideas such as limit to spend more time supporting students to understand the ideas of infinitesimals and how this can support meaningful understanding of rate of change functions and accumulation functions. This curriculum was designed explicitly to support students in developing rich mathematical meanings, and is thus inherently responsive to how students think about calculus and what today’s students should be learning in a calculus course. As currently taught, I witnessed this curriculum equitably engaging a racially diverse student population in rich mathematics. This curriculum could go further in the future by engaging the diverse learners as whole people, by situating the mathematical content in contexts that are especially interesting and relevant for them (where these contexts could be identified by talking to students and using local data to identify trends in women and students of color’s majors).

### **Shift in Pedagogy**

Through PtC, I observed three DIRACC calculus courses at LSU, and though the three courses looked different, in each I witnessed a racially diverse group of students equitably engaging in rich mathematics, contributing to constructing mathematical meaning as a class. In one class, the instructor stood in front of a 40-person class, while he randomly selected students to answer questions related to a context problem they worked on. The questions he asked were substantive and open ended, allowing every student to contribute thinking related to the question rather than simply answering correctly or incorrectly. The second class was a 120-person class where the instructor presented a slide presentation wearing a microphone, with three Learning Assistants circulating the room, and students discussing problems in small groups. The third class was a 30-person class where students spent the entire class working in groups of three-four on rich tasks while the instructor floated around the room, visiting with individual groups, and then bringing the class together for a whole-class discussion. The common element of these courses, in addition to the content being taught, was that the instructors authentically cared to understand what their students were thinking related to the mathematics, and that the instructors used this understanding of their students’ thinking to connect

the mathematics to the students' understanding of the mathematics – what Hackenberg (2005) has called exhibiting mathematical caring relations.

A forward thinking calculus program should be developed so that calculus courses, including the class-time, course topics, and out-of-class assignments, are designed to encourage a diverse set of students to succeed in the course as well as in courses building on calculus and in their STEM careers. The DIRACC curriculum and its enactment at LSU illustrate such a program by centering the mathematics, and every individual student's construction of the mathematics, as the guiding forces.

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# How professional obligations can help understand decisions in the teaching of calculus across institutional contexts

Patricio Herbst<sup>1</sup> and Mollee Shultz<sup>2</sup>

<sup>1</sup>University of Michigan, United States of America; [pgherbst@umich.edu](mailto:pgherbst@umich.edu)

<sup>2</sup>University of Michigan, United States of America; [molllee@umich.edu](mailto:molllee@umich.edu)

There is a scarcity of analyses of teacher decision making in the context of calculus instruction. We promote such a discussion, motivated, on the one hand on the breadth of considerations of the practical rationality framework (Chazan et al., 2016) and on the other hand on the diverse institutional manifestations of calculus education in the U. S. Calculus is taught in US universities, two-year, community colleges (CC), and high schools (HS). These diverse contexts are tied to a diversity of roles calculus courses play for institutions and for students, which creates conditions and constraints for the teacher's work. It is likely that the work of the teacher teaching calculus will be influenced by considerations that go beyond the discipline and its applications, or the students' learning resources and difficulties. How can we conceive of an agenda for research on the teaching of calculus that can be prepared to attend to the various considerations a teacher needs to make?

## Practical rationality

Practical rationality seeks to understand the rationality behind what teachers do. It postulates that teachers' actions result from a sensibility and can be understood as reasonable outcomes of a teacher's adaptation to their conditions of work. Practical rationality adds to usual considerations of teacher agency and knowledge, explicit attention to various types of context in activating such agency and knowledge and in providing sources of justification for possible decisions. Two important constructs, helpful to to understand these different types of context and create instruments to gauge their influence are the notions of instructional norm and professional obligation.

The notion instructional norm, alludes to resources and constraints available in the work of instruction—the work of transacting target mathematical knowledge and mathematical work with students. The notion of norm asserts that there are some regularities in the communication of knowledge in which teacher and student are involved. Some of those regularities issue from the global didactical contract that, for example, binds the instructor and the students to a course of studies such as calculus (e.g., the notion that if the calculus instructor poses a real-world problem to students, they are expected to make use of what they know from the class to solve the problem). More specific regularities concern what Herbst and Chazan (2012) call instructional situations. These situational norms bind instructor and students in regard to the specific mathematical work to be done (e.g., the separation between the work one does informally to find an appropriate value for  $\delta$  and what one writes or shows, when using a value of  $\delta$  to prove a limit by definition).

The notion of professional obligation, alludes to resources and constraints available for a mathematics teacher employed in an educational institution—hence hired to play a role in a range of activities, one of which is instruction. Herbst and Chazan (2012) name four of these obligations: to the discipline of mathematics, to individual students, to society and social groups such as a class, and to the institutions

that make room for instruction. In this paper we illustrate how the obligations, might help understand the different professional positions of calculus instructors. To do that we focus on one important institutional distinction in the teaching of calculus in the United States.

## **The teaching of calculus in the United States**

In the United States as in many other countries, Calculus is taught both in colleges and in HSs. Champion and Mesa (2018) report that 19% of all HS students in the US take Calculus. These mostly affluent students do so in the context of the Advanced Placement (AP) program, which consists of a yearlong course culminating in a standardized exam that may provide students with college credit. Offering AP courses is usually a marker of prestige for HSs and having taken AP classes is a marker of ambition for students in college admissions. Around 43% of those students (or 8% of all HS students) score 4 or 5 in the exam, hence actually receive college credit. In HSs, being asked to teach calculus is an aspiration for the best trained and most ambitious teachers. Likewise, in two-year CCs, the teaching of calculus is a small operation and the assignment to teach those classes is reserved for instructors with experience and prestige (Bressoud, Mesa, & Rasmussen, 2015). In contrast, while only a minority of students take calculus in four-year colleges and universities, the usual calculus sequence is a prerequisite for studies in science, technology, engineering, and mathematics. The teaching of calculus in US universities is the largest operation carried out by mathematics departments, usually employing a variety of instructors, tenure stream faculty as well as temporary lecturers and graduate students (Hagman, Johnson, & Fosdick, 2017). So, while the content of instruction in calculus courses may bear important resemblances across contexts, the institutional context in which the courses are situated and the position of instructors may vary.

We believe the practical rationality framework may be useful in ascertaining these differences. To illustrate this, we bring in items from an assessment that we have developed to gauge the extent to which instructors recognize the professional obligations of mathematics teaching defined above. We speculate how the recognition of these obligations may differ across the three sites noted above.

## **Instructors' recognition of the professional obligations of mathematics teaching**

The PROSE (Professional Obligations Scenario Evaluation) instrument presents instructors with short classroom scenarios in which an instructor departs from the plan of the lesson to attend to an issue related to the various obligations. It asks respondents to indicate their agreement with the decision the instructor has made. We use examples of these items to illustrate how institutional context may matter in the extent of instructors' agreement with those decisions.

### **Interpersonal Obligation**

Consider item A3145, which presents an instructor who is teaching about implicit differentiation and provides an example related to the speed of the mallet of a polo player. Some students are engaged discussing the example, but upon realizing that other students don't know the game of polo, the instructor changes the example to one of a moving observer who sees a car approaching. Respondents are asked to indicate how much they agree that the instructor should have continued discussing the initial example instead of changing it. The item represents one way in which the *interpersonal obligation* (to society and social groups) might impact instructional decisions: The depicted instructor

appears to have tried to repair social class differences associated with knowing polo, an affluent people's game. We conjecture that an item like this might generate different responses for HS calculus instructors and for CC calculus instructors (whose students tend to come from blue collar backgrounds). Also representing the interpersonal obligation is item A3025. A student repeatedly volunteers input on how to solve an optimization problem on fencing an area of land. Respondents are asked to indicate the extent to which they agree the instructor should continue to encourage the one student's participation, rather than seek a response from a different student. The interpersonal obligation is at stake because the instructor's choices affect what type of participation is fostered among the students. We expect that HS calculus instructors would be more inclined than college instructors to closely monitor student participation, as the latter might be more likely to think of students as adults who can claim space on their own.

### **Individual Obligation**

Item A4095 exemplifies the individual obligation. An instructor asks students to use the definition of the derivative of a function to find the derivative of  $g(x) = x^2$ . A student eagerly says that he has learned from someone else a quicker way of calculating the derivative of that function. The instructor asks the student to come up to the board and show his method. Respondents are asked to indicate the extent to which they agree that the instructor should have ignored the eager student. This item was created to represent a way in which the *individual obligation* (to students' cognitive and emotional needs) might impact instructional decisions. We expect that, while all respondents might be keen to see an individual student who is eager to participate, university instructors might be less inclined than CC and HS instructors to make room for such enthusiasm for shortcuts before students know how to use the definition of derivative. CC instructors are known to prioritize giving students a sense of achievement and self-confidence at the cost of giving novel, challenging mathematical tasks (Mesa, et al., 2014). Meanwhile, HS instructors might be more concerned with preparation on standardized tests than with the mathematical justification of techniques.

### **Disciplinary Obligation**

Consider item A1125 in which a calculus instructor has been discussing some problems about geometric series. After finishing a problem, he second guesses his plan to go on to the next problem and announces that he would like students to find for which ratio a geometric series would have a sum of  $3/2$ . Respondents are asked to indicate the extent to which they agree that the instructor should have continued with the next problem instead of asking a question that was outside the plan. This item was created to represent a way in which the *disciplinary obligation* (to the discipline of mathematics) might impact instructional decisions, as high recognition of this obligation would justify holding off the next problem to ask a mathematically interesting question. We anticipate that HS instructors, who are accountable to follow the AP preparation program, would be more likely than college instructors to expect the instructor to go to the following problem. College instructors might also have more liberty to pose problems that they find mathematically engaging.

### **Institutional Obligation**

In item A2115 an instructor says that as they are done with implicit differentiation, they will start with related rates. Students ask him to review a problem on implicit differentiation, but he says that

as the other sections of the class are already past related rates, they have to catch up. Respondents are asked to indicate the extent to which they agree that the instructor should have answered the questions the students had instead of moving on to the next topic. The item represents how the *institutional obligation* (with the pacing chart) may impact decision making. While individuals who have a high recognition of this obligation would be expected to support the decision of the instructor in the scenario, we also expect college instructors of calculus courses taught for many sections with coordinated exams would be more sympathetic with the decision made in the scenario than other instructors would. We expect that college instructors might solve this dilemma by offloading the question to office hours, which HS teachers do not have as a resource.

## Conclusion

The eight items discussed illustrate a span of considerations to justify departures from instructional norms that may be justified by the professional obligations postulated. The items also illustrate how the obligations and the instances for their application can reveal aspects in which the context of calculus instruction varies. Indeed, while the obligations apply to instructors across institutions where calculus is offered, the role that calculus plays in those institutions affects not only who gets to teach calculus but also, we conjecture, how these individuals make instructional decisions. More important than the specific responses to these items, however, is to explore decisions in which the selective nature of the group of students, the institutional structures for course management, and others such issues may explain how calculus instructors may make different decisions for similar instructional circumstances when those circumstances arise in different institutional contexts.

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# Teacher and lecturer perspectives on secondary school students' understanding of the limit definition of the derivative

Michael Jennings<sup>1</sup>, Merrilyn Goos<sup>2</sup> and Peter Adams<sup>1</sup>

<sup>1</sup>The University of Queensland, Australia; [msj@uq.edu.au](mailto:msj@uq.edu.au), [pa@maths.uq.edu.au](mailto:pa@maths.uq.edu.au)

<sup>2</sup>University of Limerick, Ireland; [merrilyn.goos@ul.ie](mailto:merrilyn.goos@ul.ie)

## Introduction

The transition of students from studying secondary to tertiary mathematics has been the subject of increasing research interest in recent years. One relatively new research area is into the perspectives of teachers and lecturers on this transition. Hong et al. (2009) compared teachers' and lecturers' perspectives of calculus teaching and learning. They found that teachers lack a clear understanding of the issues involved in the transition, there is a greater use of technology at tertiary level, tertiary institutions are better resourced, there is a more formal treatment of mathematical content in tertiary teaching, and secondary teachers interact more with their students.

Klymchuk, Gruenwald, and Jovanoski (2011) surveyed 63 lecturers from 24 countries on why they thought students found university mathematics difficult. Klymchuk et al. found that there is a lack of knowledge and awareness by university lecturers of what is happening at school, both in terms of content and teaching approaches.

In their paper on two practical aspects of the secondary-tertiary interface – teaching style and assessment – Thomas and Klymchuk (2012) found that teachers and lecturers each lack a clear understanding of the issues involved in the transition from the other group's perspective, and state there is a need for improved communication between the two groups.

Apart from Hong et al., little research appears to have been undertaken on the secondary-tertiary transition from a calculus perspective. This paper goes some way to filling this gap. It reports on one part of a larger, two-year longitudinal study on issues in students' transition from secondary to tertiary mathematics study in Australia.

## Methodology

In the first part of the longitudinal study, 750 students completed eight mathematics questions on pre-calculus and calculus topics taken from the Queensland and Australian school syllabi. The students were comprised of 470 Intermediate Mathematics (IM) only students and 280 students studying both Intermediate and Advanced Mathematics (AM). Responses were coded according to their displayed level of understanding. Teachers and lecturers were asked how difficult they thought students would find each question. Response options were: very hard, hard, neither hard nor easy, easy, and very easy, and comments could be included. Fifty teachers and 16 lecturers from across Queensland responded. In each of the eight questions, the Fisher Exact Test was used to determine whether there was a significant difference in perspective between the teachers and lecturers. This paper reports on the results for the following question, involving the limit definition of the derivative.

**Q.** The definition of the derivative (or the derivative from first principles) is given by the following:

$$\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} \text{ or } \lim_{b \rightarrow a} \frac{f(b)-f(a)}{b-a}.$$

(a) What does this definition mean?

(b) Use this definition to determine  $f'(x)$ , where  $f(x) = x^2$ .

## Results

**Part (a)** IM students ( $n=470$ ) results: 2% correct, 28% partially correct, 36% wrong, and 35% blank. AM ( $n=280$ ) results: 12% correct, 47% partially correct, 21% wrong, and 20% blank.

Teachers' perspectives were: 54% hard or very hard, 29% neither hard nor easy, and 17% easy or very easy. Lecturers' perspectives were: 81% hard or very hard, 6% neither hard nor easy, and 13% easy or very easy. There was a statistically significant difference ( $p < 0.01$ ) between the teachers' and lecturers' perspectives. Teachers and lecturers provided many comments on this question. One point of particular interest was the teacher comments regarding the intended and enacted curriculum. Several teachers said that the meaning of the limit definition of the derivative was no longer in the syllabus and therefore not taught anymore, hence students would have difficulty with this question.

They would possibly not have been asked to reproduce such a response before and the syllabus does not explicitly require students to explain the meaning of the derivative from first principles, merely to use it.

However, other comments indicated that teachers were in fact teaching the limit definition.

This is a great question. I feel few students in Queensland would be able to answer this question well. I have streamed classes with top maths students, I would be disappointed if most couldn't provide an adequate answer.

The IM syllabus, unlike the AM syllabus, is the same for all schools in Queensland. Theoretically, therefore, all students should be taught the same material. Despite what some teachers said, the conceptual understanding of the derivative is included in the syllabus. It is apparent from the teacher comments that not all teachers are teaching the same content and that there are different experiences depending on the school and the individual teacher.

There were a number of teachers who thought that students would be able to correctly answer the question. However, other teachers thought differently. Comments included:

As it is an integral part of the course I would be surprised if most of the students did not recognise it and be able to explain what it means.

Even though it's been carefully explained when calculus is first introduced, students tend to "move on" from anything like proof or in this case a fundamental concept explanation, unless we have repeatedly emphasised it. Sad. Many schools and teachers don't value the fundamental concepts - they just want action, so you don't see many questions of this type any more.

A comment from one lecturer merits further discussion:

I would normally not expect a high school student to give a reasonable response to this question because too many high school math teachers couldn't either.



As the surveys were anonymous it is not possible to explore this comment with the particular lecturer. Klymchuk et al. (2011) found that there is a lack of knowledge and awareness by university lecturers of what is happening at school, but they did not specifically ask lecturers for their perspectives on secondary school teachers. There appears to be no literature specifically on this area, suggesting a topic for future research. Certainly the lecturer quoted above appears to have a negative opinion of secondary school teachers, although it must be noted that the lecturer stated they have no formal teaching qualifications. It can be said that very few university mathematics lecturers in Australia have formal teaching qualifications.

**Part (b)** IM students ( $n=470$ ) results: 13% correct, 10% partially correct, 50% wrong, and 26% blank. AM ( $n=280$ ) results: 42% correct, 15% partially correct, 29% wrong, and 12% blank.

Teachers' perspectives were: 23% hard or very hard, 33% neither hard nor easy, and 44% easy or very easy. Lecturers' perspectives were: 81% hard or very hard, 12% neither hard nor easy, and 7% easy or very easy. There was again a statistically significant difference ( $p<0.001$ ) between the perspectives of teachers and lecturers. Nearly all of the lecturers thought that students would find this question difficult, whereas the teachers were more inclined to think that students would find this question easy. Comments from teachers (T) and lecturers (L) included:

(T) I think every Year 12 Maths B student would know that even if they knew nothing else.

(T) Some would just answer  $2x$ . Others may substitute  $x+h$  into the function & expand, collect like terms. Very few would be able to separate the fraction & show how the solution is obtained.

(T) I anticipate my students will only remember the short cuts, not how to apply the limit, but this is something most of them could attempt.

(L) I would expect year 12 students would be able to substitute the function into the definition and use simple algebra to work it out. Some students might have problems with the algebra.

(L)  $f(x + h) = x^2 + h$  or  $f(x + h) = x^2 + h^2$  or  $f(x+h) = f(x)+f(h)$ .

(L) Do you mean how *\*should\** they respond, or how *\*will\** they respond? They *\*will\** respond in exactly the same way their teacher provided in examples, which they will look through their notes for and copy. If they don't, communication will be very poor. Most students, except the good advanced maths students, will be stuck at  $(x+h)^2$ , *\*if\** they get that far.

The student answers and the teacher and lecturer comments raise two questions:

1. Why were the teachers so divided in their opinions of whether students could successfully find a derivative from first principles?
2. What are the implications of teachers and lecturers holding different perceptions of students' calculus skills and understanding?

For the first question, there may be several reasons for the difference in teacher perspectives. The teachers who thought that students could do this question easily may have worked through multiple examples using the limit definition of the derivative, or they believed their students had good algebra skills. The teachers who thought that students would have difficulty may not have spent much time

on the limit definition, or they may have thought that the amount of time that had passed between teaching the limit definition and students completing this survey was too great for students to remember. Given the low success rate in this question – 13% for IM students – it was clearly the case that the teachers were too optimistic.

For the second question, there are considerable implications for students transitioning to university mathematics and the lecturers who will teach them. Lecturers will be faced with some students who have not been taught the meaning of the limit definition of the derivative, some who have been taught it but don't remember or have poor algebra skills, and other students who do understand and have good algebra skills. Given this, how should a lecturer approach teaching introductory calculus? Options could include assuming that students have learnt and understood the intended curriculum, assuming that students do not understand the intended curriculum, or something in-between. Students are subsequently affected by the lecturer's choice.

## Conclusion

Teachers and lecturers would ideally have similar perspectives regarding students' abilities to solve particular mathematics questions. This would allow students to undertake a more seamless transition in their studies from secondary to tertiary mathematics. This study, however, has shown that the perspectives are different, both within each group and across groups. There are differences at the school level between the intended calculus curriculum and the enacted calculus curriculum, with subsequent implications for tertiary mathematics and how it is taught. Lecturers have also shown limited knowledge of student understanding and teaching approaches in schools. More discussion within and across teacher and lecturer groups is needed in order to assist students in their calculus journey.

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# The dual nature of reasoning in Calculus

Ivy Kidron

Jerusalem College of Technology, Israel; [ivy@jct.ac.il](mailto:ivy@jct.ac.il)

This paper is divided in three parts. The first part is devoted to the cognitive difficulties which are related to the dual nature of reasoning in Calculus as described in the early researches. In the second part, I analyze how more recent researches use the dual nature of the reasoning in Calculus to overcome some of the cognitive difficulties. I use the term “duality” in relation to abstract notions which can be conceived in two different and sometimes contrasted ways. These contrasted aspects are complementary and approaches which are based on an interplay between the two contrasted aspects might help overcome the cognitive difficulties that accompany central notions in Calculus. The third part relates to my present research on analogical reasoning in Calculus. An approach which used a specific abstract/concrete interplay is introduced and analyzed.

## Cognitive difficulties related to the dual nature of reasoning in Calculus

The lenses offered by the *process-object duality* highlight students' dynamic process view in relation to concepts such as limit and infinite sums, and help researchers to understand the cognitive difficulties that accompany the learning of the limit concept. Another duality, *finite-infinite processes*, characterizes central notions in Calculus. The *potential infinity/actual infinity* duality, is discussed in Fischbein, Tirosh & Hess (1979). These dualities, as seen in the early researches, enable the epistemological analysis of the content under consideration as well as the realization of the ensuing cognitive challenges inherent to the epistemological nature of the central notions in Calculus. Focusing on cognition, we consider the dual nature of the interaction between intuition and formal reasoning, in the terms *concept definition and concept image* (Tall & Vinner, 1981).

## The school- university transition

This dual nature of the reasoning in learning Calculus appears in the different stages of students' education. The dual nature of the interaction between formal and intuitive reasoning represents one source of difficulties in the school-university transition: The main source of difficulty in the school-university transition resides in the fact that many students' intuitive ideas are in conflict with the formal definition of the calculus concepts such as the notion of limit. On the one hand, the definition of the concept of limit is a particularly difficult notion, typical of the kind of thought required in advanced mathematics. On the other hand, in most countries today, at high school level, there is an effort to develop a first approach to calculus concepts without relying on formal definitions. Reconstructions from a more familiar, intuitive view to a formal approach have been proved to play a crucial role in calculus, especially at school-university transition (Artigue, 2001). The duality process/object might represent another source of difficulties in the school-university transition in relation to the concept of function. In differential equations, students have often to refer to function as an object, as a variable which will be calculated while solving the differential equation. At the school-university transition, the challenge is to deal with concepts which have the dual role of being familiar (intuitively) to the students and also provide the basis for later mathematical development (Tall, 1992).

## More recent researches on Calculus that use the dual nature of the reasoning to overcome some of the cognitive difficulties

In Kidron (2008) I present an approach based on the *discrete/ continuous* interplay. The aim of the study is to analyze the effect of this interplay on the students' understanding of the need for the formal definition of the derivative as a limit. Previous precept related researches influenced the specific discrete continuous interplay that motivated the design of the learning experiment. The previous researches demonstrated that the students viewed the limit concept as a potential infinite process. They also demonstrated students' belief that any property common to all terms of a sequence also holds for the limit. The derivative might be viewed as a potentially infinite process of  $(f(x+h)-f(x))/h$  approaching  $f'(x)$  for decreasing  $h$ . As a result of the belief that any property common to all terms of a sequence also holds for the limit, the limit might be viewed as an element of the potentially infinite process. In other words,  $\lim \Delta y/\Delta x$  for  $\Delta x \rightarrow 0$  might be conceived as  $\Delta y/\Delta x$  for a small  $\Delta x$ . A counterexample is presented that demonstrates that one cannot replace the limit, " $\lim \Delta y/\Delta x$  for  $\Delta x \rightarrow 0$ " by  $\Delta y/\Delta x$  for  $\Delta x$  very small and the limit could not be viewed as an element of the potentially infinite process.

In the following counterexample (the logistic equation), the analytical solution obtained by means of continuous calculus is totally different from the numerical solution obtained by means of discrete numerical methods. The essential point is that using the analytical solution, the students use the concept of the derivative as a limit " $\lim \Delta y/\Delta x$  for  $\Delta x \rightarrow 0$ " but, using the discrete approximation by means of the numerical method, the students omit the limit and use  $\Delta y/\Delta x$  for small  $\Delta x$ . (Kidron, 2008, p.202)

Durand-Guerrier (2016) presents a didactical situation aimed at fostering the understanding of the relationships between discreteness, density-in-itself and continuity for an ordered set of numbers at the undergraduate level. The research aimed at facing the didactical challenge of making students aware that an ordered set being dense-in itself does not mean that this set is continuous. Kidron and Tall (2015) use the same dualities (potential infinite/ actual infinite, process/object) that were mentioned in relation to cognitive difficulties, in a didactical situation that will facilitate the transition to the formal definition of the limit of sequences of functions. The students were introduced to two approaches to approximate functions by Taylor polynomials. One approach involved the students following Euler's text, aided by the Mathematica package in order to do the "continued division procedure" to calculate  $1/(1-x)$  as described by Euler's "development of functions in infinite series". The second approach used the notion of order of contact and a dynamic graphical approach, drawing with Mathematica successive Taylor polynomial approximations of degree  $n$  to see that, as the error term becomes small as  $n$  increases, the graph of the successive polynomials soon looks virtually the same as the graph of  $f(x)$ . A potential infinite process of getting closer to the function is presented in this second approach in contrast of the actual infinite sum as an object which is presented in Euler's approach. Kidron and Tall (2015) analyze the students' evolution of ideas from a potentially infinite process to the limit as an object, in terms of Euler's symbolic view of a power series as an "infinite polynomial", and the visual convergence of the finite polynomial approximations to the function itself. The study demonstrates how the dynamic blending of visual and symbolic representations might lead to the formal definition of the concept of limit.

## **The abstract / concrete interplay in reasoning in Calculus by means of analogy**

The main focus of my present research in reasoning in Calculus by means of analogy, is a micro-analytic study of the cognitive process involved in students' expansion of existing knowledge and construction of new mathematical knowledge by means of analogy. Reasoning by analogy includes intuitive mechanisms of thought that accompany students' processes of conceptual thinking. Therefore, the analysis is designed to elucidate the intricate relationships between intuitive mechanisms of thought, different types of analogy, the processes of reasoning by analogy and the emergence of new (to the learner) knowledge constructs.

### **Geometrical analogy**

The joint effect of abstract and concrete aspects makes possible the evocation of a previous situation and the emergence of the geometrical analogy (Hofstadter and Sander, 2013). This joint effect is demonstrated in Kidron (2018). Students, who did express their awareness of the existence of irrational numbers, expressed their belief that there are only rational numbers on the number line. Only those students who were able to recall how to obtain the square root of 2 on the number line and point *concretely* where this point is on the number line, were able to overcome their belief that the length of a segment should be expressed only by means of rational numbers. It is the geometric analogy that gives body to entities such as the irrational number  $\sqrt{2}$  whose existence on the number line seems counter-intuitive. The existence of abstract mathematical object such as  $\sqrt{2}$ , is legitimated by such geometric analogy by giving concreteness to the abstract object.

Another example is investigated in my present research in an activity in Calculus for first year students. Students might hold some naïve analogy such as the following one: "tending to the limit zero means steadily decreasing to zero in a monotonic way". The awareness of the limitations of the naïve analogies leads to the need to expand previous knowledge. In this example, the dynamic graphical approach is used to hand the limit concept explicitly. What is specific to this approach is that the dynamic graphical feedback includes an unexpected little jump back in a 3D animation that demonstrates how an expression (in this case the upper estimate of the error in the remainder of Lagrange  $R_n$ ) that approaches zero does not necessarily "steadily" decreases for every  $n$  in a monotonic way. The dynamic visual *perception* of the unexpected little jump back, creates a conceptual embodiment that enables a conceptual jump towards the definition of the limit notion.

### **A specific interplay concrete/abstract in different situations with the same essence**

The activity is centered on the use of the notion of one-to-one correspondence. This specific case of abstract/concrete interplay is used to enable the construction of a counter intuitive mathematical idea: The construction of the concept of equivalence of infinite sets that includes the idea that two infinite sets A and B for which A is included in B might be equivalent. Different pairs of infinite sets are proposed and the students reason by means of analogy with previous cases. In some cases, students were able to construct the one-to-one correspondence between the two infinite sets and use formally the definition of the equivalence of infinite sets but to be able to accommodate this definition requests another type of construction process: the construction of the link between the concrete existence of the one to one correspondence between the two sets (which we are able to point on it and observe directly) and the more abstract aspect of the equivalence of two infinite sets. The link is from concrete,

directly observable aspects of the one-to-one correspondence to the more hidden, abstract ones of the equivalence of the two infinite sets. Building the process which enables to use the formal definition may occur in a specific situation. To construct the definition requests, in addition, to find the same abstract essence in several different situations. This might be done with analogical reasoning. We are therefore not dealing with a transition from concrete to abstract. We deal with a specific concrete/abstract interplay that reappears in different situations with different “surfaces” but with the same abstract “essence” (surfaces and essence as used by Sander).

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# Students resolve a commognitive conflict between colloquial and calculus discourses on steepness

Igor' Kontorovich<sup>1</sup>, Rowan Herbert<sup>2</sup> and Caroline Yoon<sup>3</sup>

<sup>1</sup>The University of Auckland, New Zealand; [i.kontorovich@auckland.ac.nz](mailto:i.kontorovich@auckland.ac.nz)

<sup>2</sup>The University of Auckland, New Zealand; [r.herbert@auckland.ac.nz](mailto:r.herbert@auckland.ac.nz)

<sup>3</sup>The University of Auckland, New Zealand; [c.yoon@auckland.ac.nz](mailto:c.yoon@auckland.ac.nz)

## Background

The notion of steepness crops up in a variety of situations within and outside of mathematics classrooms. It is familiar as an experience that humans encounter everyday and discuss casually, yet it can also be quantified as gradients and visualized as tangent lines. In this way, steepness belongs to what Sfard (2008) calls the “interface between mathematical discourse and ‘real-life’ talk” (p. 226). As mathematics education calls to build on students’ real-life experiences, we consider steepness to be worthy of disciplined inquiry due to the opportunities that it brings for learning by encouraging conflicts between colloquial and mathematical discourses.

Here, we share a taster of research findings and teaching implications from our analysis of data that were collected from a larger project on undergraduates’ understanding of calculus (Yoon, Dreyfus, & Thomas, 2010). Our question was, “*How can colloquial discourses on steepness inform the development of calculus discourses of undergraduate students?*”

Students’ approaches to steepness have been relatively under-researched in mathematics education, especially compared to the vast body of knowledge on their understanding of the related notion of slope (e.g. Moore-Russo, Conner, & Rugg, 2011). Research on steepness has tended to focus on precalculus contexts, such as linear graphs and real-life objects with a fixed steepness (e.g., ski ramps in Stump, 2001). We consider steepness to be a promising venue for exploring “big ideas” in the teaching and learning of calculus, specifically, as it links with such fundamental concepts as rates of change and slopes.

## Commognitive framework

Our study is grounded in the commognitive framework (Sfard, 2008), which usage in undergraduate mathematics education research has been on the rise (Nardi, Ryve, Stadler, & Viirman, 2014). The framework regards mathematical discourses to be distinguishable through characteristic *words* (e.g., “steepness”, “gradient”) and their use; *visual mediators* (e.g., graphs, symbols) and their use; *routines* (e.g., calculating a gradient, constructing a tangent line); and *narratives* which are defined as “any sequence of utterances framed as a description of objects, of relations between objects, or of processes with or by objects, that is subject to *endorsement* or rejection” (p. 134, italics in the original).

Learning, from the commognitive standpoint, is conceived as a lasting change in learners’ discourse in at least one of the four characteristics named above. Such a change might occur as a result of resolving *discursive conflicts* – situations where two narratives seem conflicting, in the sense that an endorsement of one entails a rejection of the other. The conflicting narratives cannot coexist in the

same discourse or in *isomorphic* discourses, those in which “whatever is said in one of them, has its clear counterpart in the other” (Sfard, 2008, p. 122). On the other hand, when the narratives emerge in discourses that differ (e.g., in their rules of substantiation), the narratives are mutually independent, and there is no imperative to endorse one at the expense of the other. Sfard terms the discourses giving rise to such seemingly conflicting narratives as *incommensurable*.

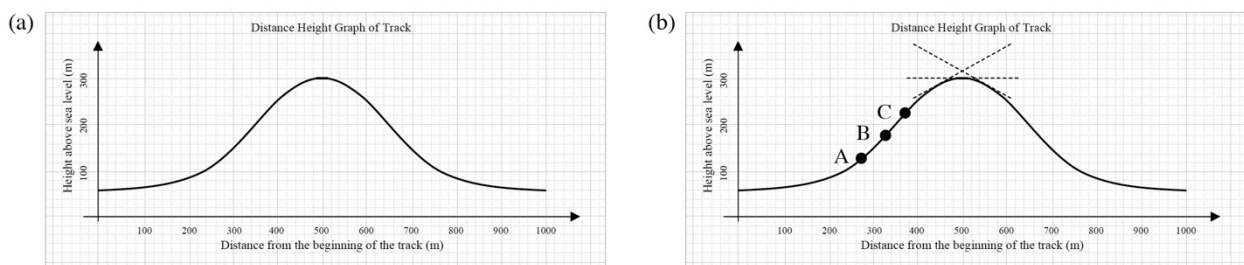
Sfard posits that some discursive conflicts are impossible for newcomers to resolve without help from the discourse’s “oldtimers”. She proposes that communication between newcomers and oldtimers can be enhanced by a shared *learning-teaching agreement* – a set of understandings regarding whose discourse is the leading one, how the teaching and learning roles are divided, and what is the nature of the expected discursive change.

## The study

The data corpus of the larger project in which our study is situated consists of video-recordings, field notes, and material artefacts that sixteen participating students created in semi-clinical sessions while working in pairs on contextualized calculus tasks. Students worked in the presence of a member of a research team, whose interaction was planned to be limited to answering clarifying questions and encouraging students to record their responses in writing, while refraining from providing mathematical aid. The plan was difficult to execute fully and consistently throughout the sessions, and the researchers found themselves shifting between non-intervening observations and overt guiding. We account for these shifts in our analysis.

The participating students were enrolled in an introductory first year calculus course at a large New Zealand university. Their enrollment indicated that the students had not studied calculus at secondary school or had not achieved highly in it. The data were collected after the topics of functions, limits, differentiation, and integration were covered in the course. The task that gave rise to the data we examine here began with a warmup, which presented students with a bell-shaped distance height graph of a tramping track (see Figure 1a) and a set of steepness-related questions.

We analyze an abbreviated exchange between two students, Adi and Lia, and the researcher Tony (pseudonyms). The exchange unfolded around the questions “Where is the track steepest uphill? How can you tell?” We chose this exchange due to the richness of the discursive conflicts that took place.



**Figure 1(a) Distance height graph of track presented to students.**

**Figure 1(b) Annotations that the students and researcher made on the graph.**



## Findings

From the beginning of the exchange, Adi and Lia focused on the upper portion of the uphill part of the graph (between  $x = 300\text{m}$  and  $x = 500\text{m}$ ). Their sketches and gestures suggest that they interpreted the word, “where” in the first question to mean a “section” or “part of the graph” rather than a point on it. When using the words “point” and “here”, the students circled sections of the graph and traced them with a pencil. Lia sketched a series of three tangents before, directly over, and after the turning point (see tangent lines in Figure 1b) and said, “Because the gradient is getting negative ... at some point it’s getting flat and then after that it is starting to get negative”. Tony asked Lia, “Where do you find in the uphill part, where do you find the steepest point, how would you look at this?” Adi responded by agreeing with Lia, “Yeah, I know what you’re saying”, retracing the sketched tangents in the air, and commenced the sentence, “It would be when...”. “Just before it’s flat”, completed Lia. Adi repeated the complete sentence as a sign of endorsement.

This episode is notable from two perspectives. First, it captures the birth of a narrative we were unfamiliar with from either educational research literature, nor from our experience of teaching calculus: that an uphill graph is the steepest “just before it’s flat”. We find this narrative peculiar not just because of its mathematical validity steepness-wise, but also because of the students’ usage of “just before” in a calculus discourse: what is “just before” a point on a continuous graph? Second, the episode is illustrative of a learning-teaching agreement that had been established in the first part of the session: Adi and Lia led the discourse, while Tony mostly sought clarifications on what was said and written. This agreement changed afterwards.

Tony repeated the peculiar narrative with a questioning intonation and continued, “If you walk up this hill [points at the graph], where would it be the hardest?” Commotively speaking, this reformulation of the original question may be viewed as giving rise to a new discourse. Indeed, while the original question was concerned with the mathematical object of “the steepest point” on the distance height graph of a track, the new discourse uses a process-oriented and personalized evaluation “would be the hardest” while inviting Adi and Lia to “walk up this hill”. A further discursive shift occurred when Tony took a ruler, and sequentially laid it on the page as if tangent lines at three points on the uphill part asking, “And that will be harder than say a bit before [the turning point] when you were here [places ruler at point A in Figure 1b] or here [point B] or here [point C]?”

We interpret Tony’s move as aiming to establish an isomorphic discourse to the students’, with the hope that Lia and Adi would attend to the conflict between the peculiar narrative that they constructed and their lived experiences. Since New Zealand students are well versed in walking uphill, there was good reason to assume that they would reject the narrative by themselves. This hope was fulfilled only partially. Lia eventually acknowledged that somewhere around the point B it “is hardest [...] because you’re walking like really up [...] and [“just before it’s flat”] you’re not walking up so much up”. Yet she added that this was only hardest “practically”, whereas “mathematically [the hardest part] will be just before you get to [the gradient of] zero”. With this sentence, Lia positioned the two discourses—practical and mathematical—as being incommensurable to one another. As “two narratives that originate in incommensurable discourses cannot automatically count as mutually

exclusive even if they sound contradictory” (Sfard, 2008, p. 258), Lia’s labeling the discourses with these different names disarmed the conflict that Tony had designed.

To complete the story of this exchange, Lia and Adi eventually rejected their peculiar narrative and wrote that the track is the steepest “from about 240m into the track to about 400m” adhering to an interval where the steepest point is included. This development occurred as a result of a coalescence of the “practical” and “mathematical” discourses, in which the students generated hybrid narratives and reapplied the ruler-mediated routine that Tony showed.

### **Implications for teaching**

To some extent, our study can be seen as a warning about the blanket approach of incorporating students’ real-life experiences into calculus teaching as a way of promoting students’ learning. The exchange we presented shows that the common pedagogical move of appealing to a discourse in which the students are better versed, does not always help them to progress in canonical mathematics. Even with pedagogical intervention, students can juggle discourses at their reach to substantiate the narratives that they created.

The exchange we presented demonstrates that students’ real-life discourses do not map neatly onto a parallel target mathematical discourse. Rather, mathematical terms like steepness (and limits, among others) belong to both discourses, each with their own accepted narratives. Thus, care needs to be taken when invoking a real-world discourse as, eventually, students are the ones to decide how this discourse connects to their classroom mathematics, if at all.

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# **Construction (and re-construction) and consolidation of knowledge about the inflection point: Students confront errors using digital tools**

Anatoli Kouropatov<sup>1</sup> and Regina Ovodenko<sup>2</sup>

<sup>1</sup>Levinsky College of Education, Israel; [anatoliko@gmail.com](mailto:anatoliko@gmail.com)

<sup>2</sup>Shenkar College of Engineering and Design and Center for Educational Technology, Israel;  
[2Regina@gmail.com](mailto:2Regina@gmail.com)

## **Introduction**

The goal of this project is an educational design to support knowledge construction, with special emphasis on known common errors (Tsamir & Ovodenko, 2013). For this purpose, we developed and designed a teaching unit that is based on a digital interactive environment. The educational goal of the unit is to construct knowledge about the concept of the inflection point in a way that will prevent common errors or will offer (to learners) tools to cope with errors if they occur. Thus, in the conference we intend to report in detail about two complementary and interlacing issues: educational design towards certain knowledge construction and the analysis of the implementation of this design.

## **Background**

The Center for Educational Technology (Israel) developed a digital teaching unit for learning and teaching the inflection point (Challenge 5, 2016). Two theoretical perspectives were interwoven to inform design: technological and cognitive. Based on the technological perspective (Naftaliev & Yerushalmy, 2011), the unit starts with the use of illustrating diagrams, continues with elaborating diagrams, and concludes and consolidates using narrating diagrams. This order allows us to lead the student from guided observation through meaningful elaboration towards a quite sophisticated concluding inquiry (ibid.). Based on the cognitive perspective, we start with the creation of the concept image that is (in our opinion) intuitively acceptable to students and is close to the formal mathematical definition of the concept. We continue, correcting misleading intuitions and gradually adding formal mathematical tools in order to achieve (as much as we can) the formal definition of the concept and its properties, as well as develop the skills necessary to distinguish between examples and counter-examples.

The unit includes geogebra labs, interactive digital questionnaires, and videos, as well as a variety of investigative assignments that are based on them. The proposed teaching unit is constituted of a series of interactive activities designed in an accessible digital environment that enables the creation of different instructional or learning processes according to the needs of the teacher and/or the abilities of the learner. The unit is intended for all students studying differential and integral calculus in high school or in their first years of undergraduate education. The unit consists of five pages (five scrolling screens), each of which includes several digital tasks (“questions,” in the terms of the unit interface). Most of these tasks are interactive (with options to check answers and receive feedback). We hypothesized that learning with this unit would allow students to confront errors and to construct (and re-construct) and consolidate knowledge about the inflection point.

With the purpose of testing this conjecture, we conducted a short feasibility study with a pair of first-year students (Gal (G) and Shani (S)) from the Industrial Engineering College (Israel). These students are considered advanced students (based on high formal achievements and their lecturer's personal opinion). It was suggested to the students that they learn the unit after they learned the concept of the inflection point during the Calculus 1 course. The students' main previous experience with the inflection point was during the investigation of functions based on the technical-algorithmic usage of well-known relevant theorems. The study was organized as a two-hour clinical interview in laboratory conditions. The students' work was documented and transcribed for the purpose of analyzing their learning process. This analysis was conducted using "Abstraction in Context" (AiC) as developed by Hershkowitz, Schwarz, and Dreyfus (2001) as a theoretical framework and, in particular, as a methodological tool using the RBC (recognizing-building with-constructing) model. In the next section we discuss the final episode of our work with the students and offer a brief analysis.

### **The episode: Judging wrong statements and inventing counter-examples**

This episode is based on the last activity of the unit, which deals with the process of the construction of the inflection point concept. At this point we expect that all the intended elements (identified from our a priori analysis) of relevant knowledge (e.g., monotonicity, tangent, derivative, convexity, etc.) have been constructed. At this stage there is an opportunity to consolidate the concept with an investigation task that presents typical errors regarding the concept of the inflection point as a collection of statements. Students are asked to determine whether a statement is true or false and, in the case of an error, to present a counter-example in the interactive narrating diagram. In this episode, to represent a constructing process (using the RBC model) we refer to the statement: [f:  $\mathbb{R} \rightarrow \mathbb{R}$  is a twice differentiable function. If the function is monotonic increasing in the neighborhood of  $x_0$  and  $f''(x_0) = 0$ , then  $(x_0, f(x_0))$  is an inflection point].

The students started with the interpretation of the given information in the statement and tried to connect their conclusions to the existence of an inflection point. The students realized quite quickly that the statement is incorrect and understood that they need the counter-example to refute the statement. But it was not easy. The first example was the (improper) counter-example  $f(x) = x^3$ .

The following episode is evidence of: (1) making a connection between the constructs; (2) making a correct judgment of the statement and the need for a counter-example; (3) recognizing (with digital help) that  $f(x)$  is not a counter-example.

S: OK, let's break this down: If in the neighborhood of  $x_0$  the function is monotonic increasing, in this neighborhood  $f'(x) > 0$  always. Does it follow that from  $f'(x) > 0$ ,  $f''(x_0) = 0$  there is an inflection point?

G: Correct. Do we need to give an example? [G. is drawing the function  $f(x) = x^3$  on the diagram] Aaaah, this is right! [...] Maybe we need a different example?

S: I think that this example meets all the conditions of the statement [reads each condition and confirms that the function fulfills it] and the point is an inflection point. We need to find a counter-example [repeats the conditions of the statement again].

In this discussion we can see some signs that fit with recognizing some elements from the students' previous knowledge (e.g., the connection between the function and its derivative, the role of the first derivative, etc.), but still there is no connection to the task context. There seems to be a circular discussion around  $f(x) = x^3$ . We can't see any evidence for building-with. It seems that finding the counter-example to this specific statement is too difficult for the students. However, an explicit prompt by the researcher "to go back" from the second derivative to the function brings immediate results. The following episode is evidence of the transition from building-with (B) to construction (C).

(B) S: OK,  $f''(x) = x^2$ . Let's go back!? But then  $f(x)$  will be not monotonically increasing. It will be  $x^4$ ? [Thinks and observes the situation with the diagram] [...] But we can force the [first] derivative to be always positive. The function  $3x^2$  is non-negative [second derivative] and if we take  $C > 0$  [first derivative] then the function will be monotonically increasing for every  $x$ .

(R) G: Why do you take  $C > 0$ ?

(C) S: Because when we get a positive derivative we get a monotonically increasing function. This is the counter-example!

(B) G: [Calculates the integrals] ... and then the next integral is  $\frac{x^4}{12} + 4x$ .

(C) S: Leave the division by 12. Take one step back. There is  $\frac{x^3}{3} + C$ . Now if we will succeed ... Look, in our first example we took a function that has an inflection point (0, 0).

(B) G: Aaaah, plus something!

(C) S: That's it! Let  $C=2$ , then we get  $f'(x) > 0$  [in the neighborhood of  $x=0$ ] and then  $f(x)$  is monotonically increasing. But  $x=0$  is not an inflection point. [G. checks the function on the diagram – Figure 1]

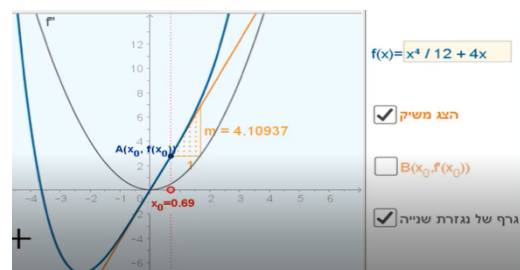


Fig. 1

It seems that while working, S and G constructed the concept of the inflection point, but apparently the construction is a fragile one. The clarification question of the researcher to G—"Can this function have an inflection point at  $x=0$ ?"—evoked doubt and put her back to the beginning of the unit. The following episode is evidence of this.

Researcher: [to G] where is the suspicious point that isn't an inflection point?

(C) G: Here [pointing to zero – Figure 2]

Researcher: Why it is here?

(C) G: Because here it's changing.

Researcher: Is an inflection point here possible?

G: Yes.

(C) S: Why? Look, the tangent line is below the graph always ...

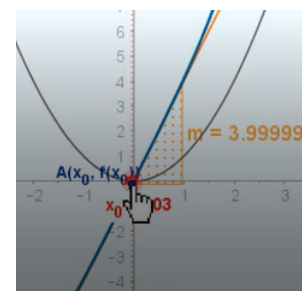


Fig. 2

G: Why? If I drag the tangent line, now it is above the graph! [Zooms in and observes] Aaaa, I see ... right! The statement is incorrect and we have found the counter-example!

It seems that G has only partially constructed the notion of the inflection point. Despite the fact that in the process of the dialogue she carried out all the algebraic operations associated with the concept, it is possible that the visual representation caused some confusion. Taking this into account, we plan to enhance the design of the diagram with the aim of supporting the constructing process of the concept.

### Concluding remarks

The analysis of students' learning process (according to the AiC framework) allowed us to gather empirical evidence regarding construction, re-construction, and consolidation of knowledge about the inflection point. Our findings show that students (partially) constructed intended elements of knowledge, (partially) re-constructed existing elements, and in some cases used constructed elements as initial elements for new construction. As a by-product of the analysis we also identified some of the disadvantages of the design. We are hoping to report and discuss the matter in a more detailed presentation at the conference.

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# Designing textbooks with enhanced features to increase student interaction and promote instructional change

Vilma Mesa<sup>1</sup>, Yannis Liakos<sup>1</sup>, and Matt Boelkins<sup>2</sup>

<sup>1</sup>University of Michigan, USA; [vmesa@umich.edu](mailto:vmesa@umich.edu) and [iliakos@umich.edu](mailto:iliakos@umich.edu)

<sup>2</sup>Grand Valley State University, USA; [boelkinm@gvsu.edu](mailto:boelkinm@gvsu.edu)

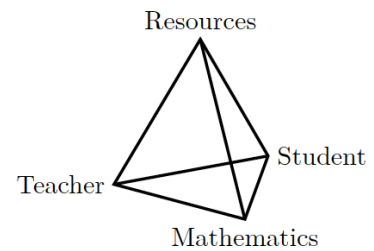
Evidence from the National Study of Calculus in the United States indicates that student engagement in calculus classes is low (Bressoud, Mesa, & Rasmussen, 2015). Observations of calculus lessons conducted in 18 different institutions revealed that, for the most part, the instructors present problems that emphasize competencies with calculus procedures (e.g., derivative rules), using symbolic representations only, and with little contextualization, visualization, or student participation (Mesa & White, 2017). While there is variation in terms of how these aspects of calculus instruction are enacted, in the end, most of the students observed did not seem to experience a conceptual approach to calculus, did not engage deeply with calculus ideas, and did not use multiple representations to solve problems in contextualized settings. Curriculum materials—textbooks in particular—have been seen as necessary, but not sufficient to promote different ways to teach calculus. In the U.S., the calculus reform movement in the 1990s sought to overhaul the teaching of calculus by emphasizing conceptual understanding of ideas such as rate of change and accumulation; the growing availability of graphing technology and an interest in making calculus “lively” generated instructional materials, textbooks and software that were meant to be used in classrooms to support students’ understanding of calculus. The introduction of new materials came with a push for new ways of engaging students in the classrooms (e.g., group work and classroom presentations, project-based learning, and more complex assessments) and institutional reorganization (e.g., longer class meetings, smaller class size, furniture to facilitate group work). Nearly 30 years later, it seems that several of these ideas have percolated into some textbooks, and while technology is even more available, instruction did not keep up. Calculus instruction is still a roadblock for student advancement in mathematics (Rasmussen & Ellis, 2013). Evidence from other studies, mainly in K-12 education, indicates that curricula alone cannot promote instructional change. However, the evidence is strong that, in the absence of appropriate materials, it is less likely that instructors will have an incentive to change their practice. As part of a large, federally-funded project we investigate how textbooks can be designed to support changes in instruction—specifically changes in how instructors and students interact with the content. The textbooks have built-in features that seek to motivate and require students and instructors to interact with ideas in ways that promote student learning.

## Theoretical background

Rezat’s didactical tetrahedron (Rezat & Strässer, 2012) helps in understanding the pivotal role of resource use in teaching (Figure 1). In the base of the tetrahedron are elements of the instructional triangle, a definition of instruction as the interaction among the instructor, the students, and the content.



In this model, resources are an interdependent element that modifies such interaction. In addition, the model allows researchers to attend simultaneously to students' use of the resources to learn mathematics (Students-Mathematics-Resources) and to the interdependent way in which instructors and students interact with the resources. Although initial conceptualizations of use have been proposed about textbook readers, ours is a purposeful attempt to investigate use of textbooks for teaching and learning in real time. The second strand of work that informs this study is the instrumental approach to resource use by instructors. Rabardel and colleagues have proposed that human instruments have a dual character: "they contain components from artefacts themselves, and components from users' utilization schemes" (Rabardel & Waern, 2003, p. 643). In this conceptualization, users interact with their resources in ways that are not anticipated by the designers. We seek to understand the schemes of use (operational invariants) associated with planning lessons and assigning homework.



**Figure 8: Didactical tetrahedron**

## Context and method

The data for the study presented here are drawn from a larger study that seeks to investigate how students and instructors use open-source dynamic textbooks in calculus, linear algebra, and abstract algebra courses. *Active Calculus* (Boelkins, 2018, pp., <https://books.aimath.org/ac/>) is an open source, dynamic textbook written using PreTeXt, an authoring markup language designed specifically by our development team to produce interactive online textbooks. PreTeXt captures the structure of textbooks to ease conversion to multiple other formats (<https://pretextbook.org/>). In addition to interactive Java and Geogebra applets, the textbook embeds WeBWorK exercises and author designed Reading Questions. WeBWorK is an open source, homework problem system with a massive open-source database of free exercises; when students complete the problems online, the system provides them instant feedback. Instructors can use such problems, independent of the textbook, to complement to their courses (see <http://webwork.maa.org/>). *Active Calculus* includes a subset of anonymous WeBWorK exercises that target skill development. At the end of each section, students encounter a set of questions prompting them to type their responses directly into the textbook. The aggregated responses are sent to the instructor in real time; those can be used to gauge

### 4.4.6 Exercises

1. Finding exact displacement.

The velocity function is  $v(t) = t^2 - 4t + 3$  for a particle moving along a line. Find the displacement (net distance covered) of the particle during the time interval  $[0, 5]$ .

displacement =

[Preview](#) [Submit answer](#) [Show correct answer](#)

WeBWorK © 1996-2016 | host: <https://webwork-ptb.aimath.org> | course: anonymous | format: simple | theme: math4

### 4.4.4 Reading Questions

1. Find an antiderivative for the function  $g(x) = 2 \sin(x) - 5e^x + x^7$ .

[My answer →](#)

2. Use the Fundamental Theorem of Calculus to determine the exact area that lies between  $f(x) = 4 - x^2$  and the  $x$ -axis on the interval  $[-1, 2]$ .

[My answer →](#)

3. Suppose that a tank filled with water is leaking at a rate of  $r(t)$  liters per minute. What does the definite integral  $\int_2^5 r(t) dt$  measure? Why?

[My answer →](#)

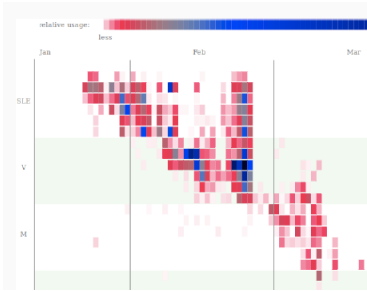
**Figure 9: WeBWorK and Reading questions in *Active Calculus*.**



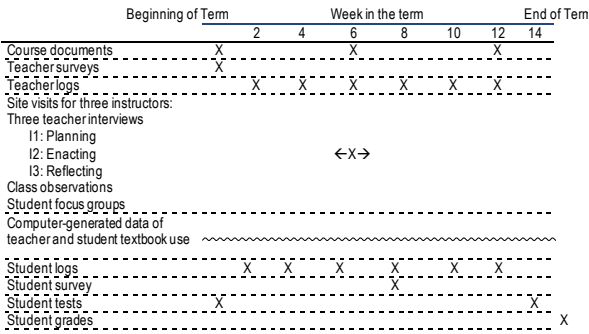
the extent to which students are understanding the material. Figure 2 presents one of six WeBWorK problems and the reading questions for the section on the “Total Change Theorem” (If  $f$  is a continuously differentiable function on  $[a,b]$  with derivative  $f'$ , then  $f(b)-f(a)=\int_a^b f'(x)dx$ . That is, the definite integral of the rate of change of a function on  $[a,b]$  is the total change of the function itself on  $[a,b]$ ).

The textbooks are distributed free of charge to the students, which reduces the cost of attending college; changes (e.g., typos, ordering, adding or removing sections or examples) can be made in real time; and because they are written in PreTeXt, it is possible to track how users interact with the textbook (Figure 3). This study involves authors, software developers, instructors, and researchers, in a continuous cycle of data collection, analysis, design, and implementation that supports the development and improvement of the textbooks and of the methods of data collection, providing a rich data set to use in understanding how instructors and students use the textbooks in the class. The study uses a mixed-methods embedded design, and collects data from the instructors, their students, and their lessons. Figure 4 shows the various types of data that are being collected over a one-semester period. Students and instructors respond to periodic surveys about textbook use (logs) that contain questions regarding their uses of the textbooks and their viewing patterns. Our pilot work has demonstrated that this strategy allows for users to reflect on what they do and why.

Our study works with 49 different sections, 17 of which use the calculus textbooks. These sections include 14 calculus instructors and their students (~280) over seven semesters of data collection. Data analysis involves both quantitative and qualitative techniques. Responses to logs and reading questions are analyzed on an ongoing basis, using topic modeling techniques (Blei, Ng, & Jordan, 2003) that identify major themes across a large corpus of data. These results are triangulated with student survey and real-time viewing data, which produce computer generated summaries of proportional viewing time by course, topic, and textbook element (WeBWorK problems, reading questions). Viewing data are mapped onto course syllabi. Log questions inquire also about the development of course documents (e.g., syllabus, lesson plans, assessments). For a subset of instructors, we perform site visits that include observation and video recording of lesson planning and enactment and discussion of video clips to obtain information about rules of actions with the documents, about decision making informing changes from plans to enactment, and corroboration of uses with students in focus groups.



**Figure 10: Real time viewing data**



**Figure 11: Data collection plan**

## Anticipated results

In our pilot study we documented instrumentation and instrumentalization processes in the context of linear algebra. In one case, an instructor indicated that linear dependence was revealed better via a “geometric interpretation in  $\mathbb{R}^3$  with more than two vectors [that were] linearly dependent”. Because the textbook did not include visualizations, the instructor embedded code in his lecture notes, which he distributed to the students, thus allowing them to visualize their own linearly dependent sets in  $\mathbb{R}^3$ . We anticipate that the readily available visualizations in the *Active Calculus* textbook will generate different instrumentation and instrumentalization processes through the Java and Geogebra applets. We also documented a progression from less to more dynamic uses of the instructor’s lecture notes and specific connections to the textbooks. Some instructors created lecture notes by hand, others created videos of themselves reading the textbooks and highlighted key points, others created electronic slides that they presented in classrooms, and still others created live interactive documents that were modified during class and became part of the students’ personal notes. With topic modeling, we identified distinct types of student engagement with the textbook (e.g., reverse engineering solutions, learning proofs) that differ by textbook used. We anticipate some novel engagements as we collect data with *Active Calculus*.

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# The place of limits in elementary calculus courses

John Monaghan

University of Agder, Norway; [john.monaghan@uia.no](mailto:john.monaghan@uia.no)

This is a position paper on the place of limits in calculus courses. The courses I have in mind are ‘elementary calculus’ courses, at school or university, without  $\varepsilon$ - $\delta$  definitions. The argument in this paper do not apply to courses on analysis. For almost 200 years, limits have been at the heart of differential and integral calculus. Modern courses take heed of this and limits are usually considered as the starting point for calculus courses. For example, in the USA, the Advanced Placement (AP) calculus course<sup>1</sup> lists its big ideas, in order<sup>2</sup>, as: limits; differentiation; integration and the fundamental theorem of calculus (FTC); series. I argue that, limits, as a big idea/topic/strand, should not have first place in this ordering. Ignoring series, as this is a short paper, I first make a case for placing differentiation before limits and then consider the place of integration. I look at the history of calculus and students’ difficulties with limits as a prelude to presenting my argument.

## The history of calculus and students’ problems with limits

Leibniz and Newton independently invented it in the late 17<sup>th</sup> century. Leibniz’ approach used infinitesimals and Newton’s fluxions. Both approaches blossomed for many years but later became a source of discomfort to mathematicians. By the turn of the 19<sup>th</sup> century attention was given to limit approaches and, by the end of that century, a ‘rigorous’ formulation of the fundamental ideas of calculus was possible. At the educational level, calculus textbooks in the 19<sup>th</sup> century increasingly adopted Leibniz’ notation ( $\delta y/\delta x$ ,  $dy/dx$ ) whilst textbooks in the 20<sup>th</sup> century increasingly adopted limit approaches that we see in the AP course. This is not to say, however, that a textbook/course that adopts a limit approach will necessarily provide a rigorous formulation of the calculus. It is worthy of note here to mention that Leibniz infinitesimal calculus was given a rigorous foundation through the invention of non-standard analysis (NSA) in 1966 and subsequent simplified forms of NSA.

It is undoubtedly the case that calculus teachers have long known of students’ difficulties with limits but this knowledge was not in the public domain until the 1970s, initially through the writings of David Tall and Bernard Cornu; see, for example, Schwarzenberger & Tall (1978). Students’ problems with limits have many sources. Dynamic process vs static objects: 0.9, 0.99, ... will approach 1 but it will never get to 1 (this problem is exacerbated by the fact that mathematics is atemporal, we don’t consider ‘time to sum’ in evaluating  $\sum_{i=1}^{\infty} 9 \times 10^{-i}$ ). Language: “I don’t really see how numbers can converge” (Monaghan, 1993). Pre-calculus mathematics: decimal notation ‘teaches us’ that any number starting ‘0.’ is less than 1, so it makes sense that  $0.\dot{9} < 1$ . Problems with limits continue beyond elementary calculus. For example, the now well documented ‘temporal order’ phenomena with  $\varepsilon$ - $\delta$  definitions, first noted by Davis & Vinner (1986), “for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  ...” is often internalised as “for every  $\delta > 0$ , there exists a  $\varepsilon > 0$  ...”. Basically, limits are hard.

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<sup>1</sup> <https://secure-media.collegeboard.org/digitalServices/pdf/ap/ap-calculus-ab-and-bc-course-and-exam-description.pdf>

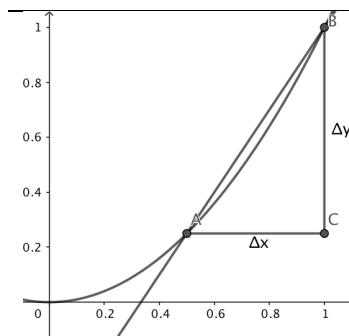
<sup>2</sup> I use the word ‘order’ to mean the order of strands, e.g. limits, is introduced before another strand, e.g. differentiation but with the proviso that the I do not expect one strand to be exhaustively covered before instruction on another begins. Most ‘good’ instruction pays attention to a spiral curriculum.

## Differentiation and integration without explicit limit arguments

But ‘limits are hard’ is only one of the grounds of my argument. Another ground is twofold: pre-calculus students have had very little exposure to limits; an introduction to differentiation gives plenty of opportunity for ‘limiting experiences’. I illustrate the last statement with a pretty standard introduction to the derivative at a point, which avoids explicit ‘limit as an object’,  $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$ , but could/should talk about the limit of the ratio  $\Delta y/\Delta x$ ,  $\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$ . The task is:

What is the gradient to the curve  $y = x^2$  at the point (0.5, 0.25)?

Figure 1 and the table to the right show how this could be approached.  $\Delta x$ <sup>3</sup> could be decremented by 0.1 and then by 0.01. The language of instruction could talk about the ‘limit of the secant’ and the ‘limit of the ratio  $\Delta y/\Delta x$ ’. The polynomial provides a relatively straight-forward context for students to ‘get a feel’ for limits whilst working on the first steps to the derivative at a point.



**Figure 1** Graphical representation of the limit of the ratio  $\Delta y/\Delta x$

$\Delta x$	$\Delta y$	$\frac{\Delta y}{\Delta x}$
0.4	0.56	0.714
0.3	0.39	0.769
0.2	0.24	0.833
0.1	0.11	0.909
...		
0.02	0.0204	0.980
0.01	0.0101	0.990

If the work is supported by a digital tool such as *GeoGebra*, then it is relatively straight-forward to change the point at which we are finding the derivative. This affords plotting the derivative at a point against this point and getting a first experience of the derivative function. This allows the derivative at a point to be fairly quickly followed up by the derivative function – something that is difficult to enact in a ‘limit approach’ to differentiation.

Traditionally, the next step is finding the derivative function symbolically. This can be done without explicitly using limits:

$$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = 2x + \Delta x$$

As  $\Delta x$  ‘gets closer and closer to 0’<sup>4</sup>, the derivative ‘gets closer and closer to  $2x$ ’. This approach can be extended to other functions and, with the help of a computer algebra system, to any function.

Without getting into the technical details of evaluating limits we can find: tangents and normals; higher derivatives; maximum/minimum points; points of inflexion; and increasing and decreasing functions. And, in covering these topics, the teacher has ample opportunity to engage her/his students in talking about limit ideas. A remarkable thing about this approach is how unremarkable it is; people do this and have been doing this for centuries. I now consider integration.

<sup>3</sup>  $\Delta x$  could be replaced by  $\delta x$  or  $dx$  or another letter.

<sup>4</sup> Another opportunity for talk about limiting processes without getting into the technicalities of evaluating limits.

Differentiation comes before integration in almost all calculus courses, but this need not be the case. 40 years ago Henle and Kleinberg (1979) wrote a calculus textbook where integration was covered before differentiation (this was enabled by the non-standard analysis approach the book took). More recently, Bressoud (2019) provides an historical account of calculus where the first two chapters are: Accumulation; Ratios of change.

In my experience, there are two common approaches to integration in elementary calculus: (i) an invocation of the FTC in the form ‘integration is the reverse of differentiation’<sup>5</sup>; (ii) a quasi Riemann-sum approach<sup>6</sup> which employs limits, e.g.  $\int_a^b f(x)dx = \lim_{\delta x \rightarrow 0} \sum_{x=a}^b f(x)\delta x$ . But there are other

approaches: (iii) through upper and lower bounds of the area under a function between two points; (iv) through a focus on accumulation and rates of accumulation (see Thompson, 1994). Approach (iii) mirrors the early history of integration (see Bressoud, 2019) but ‘post rigour’ expositions of this approach, e.g. Spivak (1967), support the approach via limit notions inherent in the concepts of supremum and infimum: a function  $f$  is integrable on  $[a, b]$  if the supremum of the lower sums equals the infimum of the upper sums (for partitions of  $f$  on  $[a, b]$ ). Nevertheless, it would be possible to develop a modified form of (iii) which does not explicitly refer to supremum and infimum. Whatever advantages and disadvantages this approach offers, it does allow integration to be presented before differentiation and theorems, simple ( $\int_a^c f = \int_a^b f + \int_b^c f$ ) and deep (If  $f$  is integrable on  $[a, b]$  and  $F$  is defined on  $[a, b]$  as  $F(x) = \int_a^x f$ , then  $F$  is continuous on  $[a, b]$ ), can be developed. Approach (iv)

will, undoubtedly, be presented during Pat Thompson’s plenary at the conference, so I bypass an exposition but make two comments: it does depend on the concept of rate of change; it is, like the approach to differentiation above, developed (by Thompson) without explicit limit concepts, e.g. “ $[V(h + \Delta h) - V(h)]/\Delta h$  ... represents the average rate of change of volume over the interval  $[h, h + \Delta h]$ ” (ibid., p.264).

## Discussion

I revisit the grounds for my ‘not limit first’ argument. I then consider what functions are suitable for elementary calculus without explicit limits. I end by considering the possible orderings of limits, differentiation, integration and the FTC.

The grounds of my argument are: limits are hard for students; pre-calculus students have had very little exposure to limits; differentiation gives opportunity for ‘limiting experiences’. The norm of instruction in pre-calculus mathematics is to build on students’ prior experiences. Ideally, pre-calculus mathematical activities should cater for limiting experiences but this is not, in my experience, the norm. I have pointed out areas where non-explicit limit ideas come into differentiation above but they can come into integration too, in approach (iv) and a modified approach (iii). This paper is a

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<sup>5</sup> This approach is, in my opinion, simply bad mathematics.

<sup>6</sup> ‘Quasi’ because the width of strips are assumed to be constant and only one of lower/upper rectangles are used.

position paper. If this position was adopted, then we could be designing curricular activities that generate good conditions for students' limit experiences prior to being explicitly introduced to limits.

The functions I regard as suitable for such non-explicit limit calculus are those functions we can draw in a quick movement of the hand – not just continuous but where left derivatives at a point always equal right derivative at that point. These are the functions for which limits are not particularly problematic (other, of course, than  $0/0$ ). Not only that but it would be rather silly to consider the limits of these functions at a point as it is just the value of the function at that point. Students must, of course, see other types of functions but examining limits in these other functions ( $f(x) = |x|, g(x) = \frac{x^2-1}{x-1}$ , etc) should be delayed until students have had other limit experiences.

I close by considering the order of limits, differentiation, integration and the FTC. I have argued that differentiation should come before limits are explicitly considered as mathematical objects. Integration could come before differentiation but this would require a non-standard analysis approach or a modified approach (iii) or an historical approach (Bressoud, 2019). I am not aware of curriculum development or research into starting with integration at the level of elementary calculus, so further consideration/experimentation would be required. I now move on to consider limits and the FTC. Is there a best ordering with these? I think the answer is straightforward, either can come before the other. In an AP-like approach limits will come before the FTC. But the work of Pat Thompson shows that the FTC can be taught and understood by students without an explicit consideration of limits<sup>7</sup>. Further to this Thompson (1994, p.167) contrasts his approach to a classical limits proof:

The problem with the typical proof is not so much in the proof as that it is presented as modelling a static situation. It is presented in such a way that nothing is changing. If students are to understand  $F'(x)$  is a rate of change, then something must be changing

So, instead of coming first in this calculus quartet, limits could come last. But if they come last, then do we need them at all? I end this position paper by suggesting that we can design and successfully teach elementary calculus courses without any explicit considerations of limits as objects.

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<sup>7</sup> Though this may only be practical with the use of mathematical software; which does not seem to be a problem in 2019.



# First-year engineering students' reflections on the Fundamental Theorem of Calculus

Hans Kristian Nilsen

University of Agder, Norway; [hans.k.nilsen@uia.no](mailto:hans.k.nilsen@uia.no)

## Introduction and rationale

This paper builds on a study aiming to investigate how calculus students, in their first year at university, reflect on the Fundamental Theorem of Calculus (from now on denoted as FTC). As a part of MatRIC's (*Center for Research, Innovation and Coordination of Mathematics Teaching*) research activity, the study was carried out with engineering students from the University of Agder. As the university teacher of these students explicitly emphasized the importance of obtaining in-depth understanding of fundamental concepts and ideas of integration, a relevant object of the research is to investigate if this could be traced in students' reflections. Recent research on students' perceptions of integration has inspired this study (Thompson & Silverman, 2008; Bressoud, 2011; Ely, 2017; Wagner, 2018) and I draw on the findings from these in the discussions. For this paper, I pose the following research question: *What characterizes first-year engineering students' reflections on the Fundamental Theorem of Calculus?*

## Theoretical framework and research methods

From an institutional perspective "the relation of an individual to an object,  $o$ , of knowledge is strongly conditioned by the institution  $I$ " (Winsløw, 2013, p. 2478). The object,  $o$ , in this case corresponds to the FTC. Further, within the Anthropological Theory of Didactics, a "praxeology" serves as a unit from which one can analyse human action, composed by the parts "praxis" and "logos" (Chevallard, 2006). "Praxis" consists of a certain type of tasks and a set of "techniques" to solve them, while "logos" contains two levels of descriptions and justifications of the praxis. The first level is "technology" and concerns the discourse of the techniques. The second level is the "theory" which provides the basis for the technological discourse. In this paper "mathematical praxeology" refers to the content of the lectures while students' "interpretation of praxeology" involves students' meaning-making related to the mathematical praxeology they are a part of. Winsløw (2013) describes praxis as the "practical block" and logos as the "theory block" and since technology and theory sometimes is inconvenient to separate, I draw on these terms in the analysis. A mathematical praxeology normally involves both the practical block and the theory block. However, in some occasions, for example when conducting mathematical proofs, theory could play a major role also in practical block, and proofs then "shift from the logos to the praxis" (Winsløw, Barquero, De Vleeschouwer & Hardy, 2014, p. 101).

To account for the mathematical praxeology related to the FTC, I observed two introductory lectures, each lasting for two hours. A video camera was used to record the lectures. The class consisted of mechatronic and computer engineering students divided into small "study groups". Based on voluntariness and gender balance, four such groups were selected and 15 students in total, were interviewed. The interviews were semi-structured, each lasting for about 40 minutes. All the students were asked to describe, in their own words what an integral is, followed by the challenge of explaining

the content of the FTC, as display in their textbook. Follow-up questions were based on the students' own statements. In the analysis, these two parts of the interview are treated holistically, aiming to account for the students' interpretations of the associated mathematical praxeology.

## Mathematical praxeology of the lectures

In this section, the mathematical praxeology through two introductory lectures is accounted for. In the first lecture, two types of tasks were dealt with and the first was “estimate the area between  $f(x) = x^2$  and the interval  $[0,10]$  by using  $n$  participations”. The technique used to solve this was by applying the summation formula  $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$  for the expression  $A = \sum_{i=1}^n \frac{1000}{n^3} \cdot i^2$ , for different values of  $n$ . The associated technology involves identifying the area of a random “bar” being  $A_i = b_i \cdot h_i = \frac{10}{n} \cdot \left(\frac{10}{n} \cdot i\right)^2 = \frac{1000}{n^3} \cdot i^2$  and hence the sum. Theory sustaining the technology was tacit in the lecture but involves justification of the summation formula and linearity rules for final sums. The second type of task was to “find the area as  $n$  goes to infinity”, within the same context as the previous type of task. The technique in this case was to evaluate the formula derived from the expressions in the previous type of task, namely  $= \frac{1000}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{1000}{n^3} \cdot \frac{2n^3+3n^2+n}{6} = \frac{1000 \cdot 2n^3}{6n^3} + \frac{1000 \cdot 3n^2}{6n^3} + \frac{1000 \cdot n}{6n^3}$ . By evaluating each of these three last terms as  $\rightarrow \infty$ , one is left with  $A = \frac{1000}{3}$ . The associated technology is to identify the proper summation formula based on  $A_i$  as a “random bar” of the Riemann sums. In addition to the justification of the summation formula, definition and characteristics of “limit” as a mathematical term, tacitly constitutes the underlying theory. The second lecture and the third type of task exemplified, in line with Winsløw et al. (2014), how theory could “shift from the logos to the praxis”. The type of task was not explicitly stated in the beginning of the lecture, but tacitly permeates the content, and could be formulated as “define the definite Riemann integral”. Techniques in this case was to create upper and lower boundaries, constituted by sums of “bars” respectively exceeding and descending the graph. It was concluded that the area corresponded to a number “between the lower and upper boundary of sums of bars, for all possible partitions”, which again led to the definition of the Riemann integral in terms of  $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \cdot \Delta x_i = \int_a^b f(x) dx$ . This abstraction is rather complex and belongs to the theoretical block, involving both technology and theory: Technology in the sense that it explains the purpose of constructing the sums and theory as it also rests on the ideas of infinitesimals and limits. The fourth type of task, “prove the fundamental theorem of calculus”, was explicitly stated either, but as for the previous task, it formed the basis for the content. The technique was carried out by sketching a graph,  $f(t)$ , followed by marking the integrals  $F(x)$  and  $F(x+h)$  in terms of the corresponding areas. By constructing the expression  $F(x+h) - F(x) \approx f(x) \cdot h$ , and by letting  $h \rightarrow 0$ ,  $F'(x) = f(x)$  is achieved. The technology consisted of identifying the visual representation as the definition of the derivative, while the mean value theorem, and the concepts of limits and differentiation, tacitly constituted the theory.

## Analysis of students' reflections

Since it is impossible to provide nuanced analysis from the transcriptions within the limitation of this paper, the main findings are presented in a table (Table 1). Based on “area under a curve”, which was mentioned by all the students during the interview, two main categories of further reflections emerged. “Riemann sums” alludes to reflections about sums of rectangles (or “bars”). If the students



somehow reflected on the integral as the limit of such sums, they fulfilled the criteria of an associated mathematical justification, and were listed in the theory block. If they only demonstrated the awareness of such sums and how to calculate them, they are listed in the practical block. “Anti-derivatives” includes students explaining that differentiation and integration are inverse operations, and that the anti-derivative could be used to calculate the integral. To be categorized in the theory block, the students had to provide a mathematical justification for why this is the case. The column “None” are students that only provided pragmatical answers like “I don’t know, I just do it”.

Number of students	Area under a curve				
	Riemann sums		Anti-derivative		None
	Practical block	Theory block	Practical block	Theory block	
3					X
2			X		
2	X		X		
3	X	X			
4	X	X	X		
1	X	X	X	X	

**Table 1: Students’ distributions related to their interpretations of the mathematical praxeology**

The emphasis on Riemann sums in teaching might explain that eight students are associated with the theory block of “Riemann sums”. Still, most of these students hesitated when challenged to provide more detailed explanations. One student suggested that “this is only the sum of heights [heights of the bars] with no width” while another student expressed frustration, as he commented that “it stresses me that very small pieces are an estimate, but when one approaches zero it suddenly becomes accurate”. The statement “when you calculate integrals [in substitution] you treat these as fractions, but on the other hand these are not really fractions”, illustrates a third student’s concern related to the notation  $dx$  and its implications. This might relate to the university teacher’s lecture about substitution (not accounted for in this paper) and his response to a question on whether  $du/dx$  is a fraction: “you can look at these  $[du/dx]$  like that. It works. But if you regard it from a theoretical perspective it is not quite the same”. Subsequent, “extension of the number line in terms of infinitesimals” were briefly mentioned as a rationale, but without any further elaborations.

Concerning the anti-derivative, several students point out that integration and differentiation are inverse operations, but only one student was able to offer a mathematical justification for this correspondence: “I think that if you for example have  $f$  of  $x$  equals two, then that will be a horizontal line. And when you take the anti-derivative, then you get two  $x$ . So that is only two times the width”. By using  $f(x) = 2$  as an example, the student related the anti-derivative of this function to the area of a corresponding rectangle. Due to the formulation “two times the width” it is likely that the student links this to the “bars” in Riemann sums.

## Discussions and conclusion

Compared to a more traditional cognitive approach, aiming towards the individual students’ conceptions, I find that ATD allows for institutional interpretations of students’ reasoning. By emphasizing the connection between students’ reasoning and the actual teaching they have been

exposed to and the types of tasks they have worked with, I also see more direct opportunities for improving teaching.

Demonstrated through the mathematical praxeology of the lectures, limits and infinitesimals are not treated on a meta-level, in terms of being objects for discussions and reflections themselves. Hence, one can suspect such discussions to be dismissed and considered to be unimportant for these group of students. In turn, this might also explain some of the students' reluctance and confusion related to the algebraic sense-making of notations, represented through the examples provided in the previous section. Ely (2017) points out that in most textbooks,  $dx$  and  $\int$  are still used, but without the meanings Leibniz assigned to these. Instead, praxeologies reformulate integrals in terms of limits. The notations in some sense then become vestiges and no longer directly represent quantities that students can manipulate. According to Ely (2017), this ambiguity is not easy to solve, unless one introduces hyperreal numbers to substantiate the algebraic sense-making of infinitesimals. Further, from the praxeology of the lectures, one observes that Riemann sums primarily serve the purpose of introducing integrals and the FTC. To some extent, this is mirrored in students' reflections. In this sense, findings support Wagner's (2018) claim that "too many students dismiss Riemann sums as an unpleasant stepping-stone to be endured in a curriculum whose goal was really to get to the FTC" (p. 354). As pointed out by Thompson and Silverman (2008), Riemann sums bear the potential of playing a major role for the students' perception of integrals as accumulation functions, which in turn could contribute to students' understanding of the FTC. For the 15 students in this study, the neglect of integrals as an "accumulation function", enforces the suspicion that the potential of Riemann sums is not sufficiently utilized in teaching. In this respect, Bressoud (2011) suggests that if we want students to see the need for evaluating limits of Riemann sums, we ought to provide students with good unfamiliar problems involving accumulation.

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# Towards transition-oriented pedagogies in university calculus courses

Alon Pinto

Weizmann Institute of Science, Israel; [alon.pinto@weizmann.ac.il](mailto:alon.pinto@weizmann.ac.il)

## Rationale

The current literature on university mathematics education shows that there exists a divide between mathematics as experienced in school and as practised at university (Gueudet, 2008). Some scholars have gone as far as suggesting that school and university mathematics are distinct disciplines (Sfard, 2014). In calculus, key mathematical concepts such as function, limit, derivative, definition or proof, which students are familiar with from school, take an entirely new meaning at university. Thus, calculus students struggle not only to extend and refine mathematics learned in school, but also to get to know in a very different way mathematics they thought they already master. Moreover, calculus students need to re-learn the tacit rules of talking and doing mathematics in order to participate in the university mathematics discourse, which is “as far removed from what the student knows from school as a discourse can be” (Sfard, 2014, p. 200).

Nevertheless, a growing body of evidence indicates that student transition to university mathematics is typically left implicit in the instruction at undergraduate mathematics courses (e.g., Fukawa-Connelly, Weber, & Mejía-Ramos, 2017). The research literature highlights many negative consequences of the implied requirement from students to take on the responsibility for their transition processes (e.g., Dreyfus, 1999). A partial solution to this problem is sometimes given in the form of mathematics 'bridging' courses, for example introduction to mathematical proof courses. However, it is not clear to what extent students can comprehend and internalize the subtle and tacit rules of the university mathematics discourse in this setting. A different approach for supporting university mathematics newcomers, which is the focus of this paper, are *transition-oriented pedagogies* that address student transition processes directly and explicitly within undergraduate mathematics courses. The questions guiding this article are: (1) What are the characteristics of a transition-oriented pedagogy in proof-oriented calculus courses? and (2) How can instructors in these courses be supported towards assuming transition-oriented pedagogies?

## Theoretical background

There are well-documented barriers to transition-oriented pedagogies in calculus. First, it is not clear how to operationalize student transition in terms of well-defined and assessable learning objectives. Not only ways of doing and thinking about mathematics are primarily tacit, they are also not fully agreed upon within the mathematics research community. For example, while it is widely accepted that students should learn the norms determining what constitutes a mathematical proof, studies have shown that different university mathematics instructors apply different norms to determine the legitimacy of proofs (Miller, Infante, & Weber, 2018). Another barrier to transition-oriented pedagogies is that some instructors do not consider ways of doing and thinking about mathematics as legitimate content of *mathematical* courses (Schoenfeld, 1994; Nardi, 2008). Moreover, students themselves have diversified views regarding what constitute legitimate content. For example, Dreyfus (1999) noted that his calculus students often complained about the requirements in his courses,

arguing that “they should not be required to write text because they are taking a mathematics class rather than a literature class” (p. 89). Despite these barriers, studies that investigated teaching in calculus courses suggest that although student transition is typically not discussed explicitly in class, it has significant impact on teaching decisions instructors make while planning courses or lectures, presenting proofs in lectures, or assessing student work (e.g., Pinto, 2018; Pinto & Karsenty, in press). However, there is ample evidence that instructors’ attempts to convey informal mathematical content in lectures are often not productive, and in some cases may even end up confusing students (e.g., Fukawa-Connelly et al., 2017). In this paper, I re-visit two of my studies that investigated teaching practice in calculus courses and re-examine the data and the findings in order to gain insight into the pedagogical challenges that instructors face when attempting to support student transition. My analysis draws on Commognition theory (Sfard, 2008) in order to characterize the learning that takes place in calculus courses. Due to limitations of space I do not discuss here this framework, and leave it implicit in my analysis.

### **Study 1: Lack for a shared discourse for informal mathematics**

Over the last seven years, I have conducted a series of studies that investigated the content that calculus instructors try to convey, explicitly or implicitly, in their courses. One of these studies (Pinto, 2018) investigated variability in the content two teaching assistants (TAs) tried to convey while teaching different sections of the same proof-oriented calculus course. I chose this particular calculus course because members of the mathematics department identified it as one of most difficult courses for undergraduates, and as a significant milestone in school-university transition. A second reason was that the lecturers and teaching assistants in this course made considerable efforts to coordinate the teaching in the different sections. The TAs met every week to decide together on the goals and content of their lessons, and they prepared a common lesson-plan that reflected their goals. I examined two lessons that focused on the definition of the derivative. My analysis showed substantial diversity in the content the TAs tried to convey. Specifically, I found that while on the surface the TAs were discussing the same definitions and the same examples, implicitly the TAs were trying to foster different kinds of meta-level learning. In particular, the TAs identified different transition-related challenges that they tried to address, for example the transition from the view of derivative as a property of a function as a whole, to a local property of function at a point.

While some diversity with respect to the content different instructors try to convey is unavoidable and may even be desired, in this case, the TAs seemed unaware of this diversity, and presumed that the staff meetings and common lesson-plan suffice to ensure students in different sections receive comparable opportunities for learning. However, the impact of these measures turned out to be limited to object-level learning. There could be various explanations for this gap, but one that seems best aligned with the data is that the course staff lacked a shared and explicit discourse for meta-level learning in calculus. By and large, the mathematical discourse in the staff meetings was restricted to object-level learning – definitions, theorems and proofs – whereas the mathematical discourse of the TAs reflections on their goals and considerations during the interviews revolved mostly around meta-level learning. Generally speaking, the TAs’ discourse about object-level learning was more formal and standard than their discourse about meta-level learning, which tended to be colloquial and individual. The TAs had an abundance of relatively well-agreed upon words at their disposal when

communicating about object-level learning, for example: concept, definition, theorem, corollary, proof, example, procedure, algorithm. In contrast, the TAs had their own distinct vocabularies for meta-level learning, for example: mathematical thinking, mathematical skills, mathematical intuition, an appreciation of mathematics, a sense of aesthetics, practices of making sense, heuristics, working knowledge. It seems that the lack of shared discourse for the meta-level content that students are expected to learn in calculus hindered the TAs' efforts to share and bring for discussion their goals, and to coordinate the content in their lectures.

## **Study 2: Resolving tensions between different norms of proof**

The following took place in a calculus course (Pinto & Karsenty, in press): The professor, Mike, stated Cauchy's Mean Value Theorem and then informed class they will now *prove* this theorem. Mike's presentation comprised three parts. First, he suggested a naïve approach for proving, and demonstrated how this approach ends up with no fruitful result. Then, Mike reflected on this 'failure' and suggested a revised approach, which was based on an additional hypothesis. This effort indeed yielded a valid proof. Finally, Mike demonstrated how the proof he presented could be adapted into a two-line argument that does not rely on the additional hypothesis. He concluded his presentation by stressing that the first two parts of the proof he presented are in fact redundant, as the two-line argument is mathematically valid, and in fact is identical to the proof presented in the textbook; however, Mike added, this two-line proof is unintelligible. In their home assignment, the students were asked to *prove* the theorem they just learned. One of the students, whom we will call Jane, submitted a proof that included her version of the 'redundant' chain of statements Mike presented at the lecture, which included several errors that could indicate significant mis-comprehension. The remainder of Jane's proof – the concluding two-line argument – was error-free. What drew my attention to this scenario was Mike's written feedback on Jane's proof; rather than highlighting errors in the 'redundant' part of Jane's proof, Mike instructed Jane to completely remove this part, ignoring apparent issues of miscomprehension therein. Why did Mike flatly reject his own addition to the textbook, which according to him, made the proof more intelligible?

It is widely recognized that in their first proof-oriented courses, students infer norms of proof from the tacit modelling implied in the way proofs are presented to them in lectures and in textbooks, and from how their own proofs are assessed (Dreyfus, 1999). Thus, variation in the norms of proof in different pedagogical contexts in these courses makes student transition even more challenging. For example, when students are asked to *prove* a mathematical statement in a certain context they need to decide whether 'prove' means to construct a polished "textbook-like" proof, to present a proof with meta-level commentary similar to the way proofs are presented in lectures, or perhaps to present a narrative that best expresses their understanding, in their own words. A possible resolution is to urge instructors to avoid as much as possible variation in the norms of proof, so to not confuse students. However, an analysis of Mike's goals for teaching proofs in calculus illustrates that variation in norms may be unavoidable and even undesirable, since different norms of proofs are better aligned with goals (Pinto & Karsenty, in press). For example, while grading proofs Mike had to decide whether to admit proofs that include 'redundant' text, such as student's reflection on their process of proving, or to adhere to the shared norms of the mathematics research community, which aim to conceal individual and contextual features, including the prover's thinking and understanding. Thus, choosing

which norms of proofs to endorse, explicitly or implicitly, is a genuine and recurring dilemma for calculus instructors. Admitting student proofs with ‘redundant’ text can help instructors assess student comprehension and provide better formative assessment, but it could also lead students to learn to write proofs that mathematicians would not accept.

## Concluding remarks

In this paper, I highlighted two transition-related pedagogical challenges in calculus: communicating explicitly meta-level content, and choosing which norms of proofs to endorse in different instructional contexts. These challenges suggest several research trajectories that can support calculus instructors in assuming transition-oriented pedagogies. First, scrutinizing the meta-level content in calculus courses, and identifying barriers that hinder communication of meta-level, towards a shared and refined discourse for meta-level content in calculus. Developing such a discourse could help calculus instructors to learn from and with their peers about meta-level learning, and to communicate in class content that is currently left implicit. Second, identifying pedagogical tensions and dilemmas that relate to meta-level learning in calculus courses, towards promoting instructors’ awareness of the impact on student transition of various teaching practices.

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# The mathematical apprentice: an organising principle for teaching calculus in the 21st century

Christopher J Sangwin

University of Edinburgh, UK; [C.J.Sangwin@ed.ac.uk](mailto:C.J.Sangwin@ed.ac.uk)

Calculus is still being taught, largely, using traditional methods-based approaches. Efforts to reform calculus have emphasised conceptual development over skill development.

Calculus has been so successful because of its extraordinary power to reduce complicated problems to simple rules and procedures. Therein lies the danger in teaching calculus: it is possible to teach the subject as nothing but the rules and procedures — thereby losing sight of both the mathematics and of its practical value. (Hughes-Hallett et al., 1994)

Their guiding principles for writing their textbook were (1) that every topic should be presented geometrically, numerically and algebraically; and (2) that formal definitions and procedures evolve from investigation of practical problems.

To test the assertions that (i) traditional teaching still focuses on skills, and (ii) students do not programme computers, or otherwise consider how technology works in parallel to calculus, I undertook an analysis of textbooks. Space here precludes an extensive selection of texts, or detailed methodology or results. Stewart (2013) was selected as representative as perhaps the most popular “calculus for mathematics” textbook, with Hughes-Hallett et al. (1994) as a reform calculus textbook. Stroud & Booth (2013) and James (2015) represent calculus for engineering students.

It is clear from all these textbooks that students are required to learn a range of traditional methods in anticipation of their subsequent use. (Stewart, 2007, p. 332) does contain a discussion of the limitations of integration in terms of elementary functions, at least acknowledging the problem exists. A similar discussion is in (Hughes-Hallett et al., 1994, p. 407). However neither James (2015) nor Stroud & Booth (2013) discuss the limitations of the traditional integration methods. These books contain rather little explicit use of technology beyond using a calculator or computer that can graph functions. In James (2015) students are encouraged to “*Check your answers using MATLAB or MAPLE*”, but these tools might be useful for finding answers in the first place. Students are not asked to programme code, rather technology is used as a “black box” function.

The ability to search online, and to use software such as computer algebra systems (CAS), changes our relationship with knowledge in a fundamental way. It is no longer necessary to master a method before the situation requiring it is encountered. Consider the *Lambert W-function*. The logarithm is the solution  $x$  of  $e^x=k$ , the Lambert  $W$ -function is the solution  $x$  of  $xe^x=k$ . I.e.  $x=W(k) \Leftrightarrow xe^x=k$ . The Lambert  $W$ -function has been studied in many areas of mathematics, over the last 250 years, and its properties rediscovered by a variety of mathematicians, (Corless, Gonnet, Hare, Jeffrey, & Knuth, 1996). I became aware of  $W$  when trying to find the solutions to delay-differential equations (Ilchmann & Sangwin, 2004). Maple returns solutions in terms of  $W$ , something at that point I did not understand. This reversal from pre-existing skills to interpreting the results creates a novel educational task and calls into question the necessity of teaching a *comprehensive* corpus of skills.

Some educators were initially positive about CAS, but CAS have proved difficult to learn. Arguments about technology, conceptual development and technical fluency, are over 400 years old. E.g. the invention of the slide rule in the 1630s generated a heated debate which has hardly progressed today. The question “Should Students Learn Integration Rules?” was considered by Buchberger (1990) who developed the *White-Box/Black-Box Principle* for using (symbolic) software. When “area  $X$ ” (e.g. symbolic integration) is new to the students, “*the use of a symbolic software system realizing the algorithms of area  $X$  as black boxes would be a disaster*”. When area  $X$  has been thoroughly studied students should be allowed and encouraged to use the algorithms available in the symbolic software systems. I do not agree with this principle. Instead, I want to propose an alternative, the *mathematical apprenticeship*, as a guiding principle for educational development. The proposal is based on the observation that traditionally, at least during the apprenticeship stage, an apprentice craftsman would make their own tools.

*A mathematical apprentice should write software in parallel to learning theory and through this earn the right to use the professional software packages.*

For example, writing a function which calculates the symbolic derivative of an elementary expression is a very simple task in recursive programming. Writing a differentiation function also necessitates discussion of the general rules, and derivatives of specific functions. For example, the following Maxima code defines a function capable of differentiating any polynomial  $ex$  in the variable  $v$  written in any form.

```
mydiff(ex, v):=block( if freeof(v, ex) then return(0),
  if atom(ex) then return(1), /* The only atom not free of v is v! */
  if op(ex)="+" then return(map(lambda([ex2], mydiff(ex2, v)), ex)),
  if op(ex)="*" then return(block([A:part(ex, 1), B:apply("?", rest(args(ex)))],
    mydiff(A, v)*B + A*mydiff(B, v)
  )),
  if op(ex)="^" and freeof(v, part(ex, 2)) then
    return(block([A:part(ex, 1), B:part(ex, 2)], mydiff(A, v)*B*A^(B-1))),
  return('diff(ex, v))
)$
```

This function includes the chain rule for exponentiation, linearity of addition and the product rule. Further rules can be added in due course. Edge cases, such as  $f(x)=|x|$ , could be included depending on the group. Mathematicians will be interested in foundations (e.g. limits and how these are used for individual functions) and completeness, engineers might not be so interested in these. Writing a function which “simplifies” the resulting expression is much harder, so students would write `mydiff` but use an existing `simplify` as a first pass. At some point the student might want to consider “simplification” as well. Students are going to use software: it is ubiquitous. We need students to both use software effectively, and we need a smaller group of students to maintain this software and further develop it. This principle applies to both pure mathematical specialists, and to service courses for students on engineering degrees, although the details of the tasks suitable for each group will differ according to context as appropriate.

Calculus itself is changing, with a much greater emphasis on numerical methods and an increasing acceptance of the legitimacy of numerical (rather than closed form analytical) solutions. Users of contemporary calculus place a much greater emphasis on simple programming using packages such



as MATLAB and Maple. Acheson (1997) provides a succinct manifesto for calculus at the start of the 21st century, including code for students to programme themselves.

A further major disconnect is that problems in traditional calculus courses are designed for students to do by hand. *“Tables of integrals are very useful when we are confronted by an integral that is difficult to evaluate by hand and we don’t have access to a computer algebra system”*. (Stewart, 2007, p. 328). When will a professional calculus user have access to paper-based tables of integrals but not have access to a simple computer, or internet access? CAS are capable of running on very modest hardware, on Android (i.e. a phone), and are open source. Using tables of integrals do generate an appreciation of algebraic form (e.g. to match up the completed square in the denominator of a partial fraction). While these skills are likely to be harder to develop without the requirement of doing integrals by hand, the premise that students won’t have access to a CAS is unlikely to be satisfied in any realistic scenario.

We do not teach the theory or practice of the algorithms upon which current technology relies. No mathematics department, of which I am aware, teaches the Risch algorithm for symbolic anti-differentiation, Risch (1969). The mathematical apprenticeship addresses this problem, opening up the workings of the software from the start and providing those students inclined to do so moral permission to question and further develop the software itself, in a way merely using a provided package does not. This proposal necessitates a radical re-thinking of the importance of traditional heuristic methods relative to contemporary comprehensive algorithms. This radical re-thinking is the most difficult part of this proposal.

Unlike the White-Box/Black-Box Principle, the apprentice does not start with foundations. Students will use well-developed libraries long before they study the area thoroughly. Indeed, in general apprentices do not generally start with foundations. E.g. apprentice pilots are flown to a safe height and fly the plane on the first day of training. The instructor might only permit them to operate the stick while the instructor operates all other controls, letting the student focuses on one thing. Similarly, the mathematical apprentice will be given software tools and use some existing packages to support the development of their tools. For example, writing a floating-point library will *not* be the first step, but they will eventually want to understand and implement the issues discussed in (Goldberg, 1991). In fact, the difficulty of representing continuous real quantities in a finite state machine is a central issue in real analysis. Simplification of symbolic expressions should be available at the outset, and students will start by using these, probably without noticing or questioning them too much.

The mathematical apprentice is a deeply constructivist approach to teaching. Making tangible things (even in code) can be satisfying and motivating, as well as challenging and frustrating. With careful design a student’s code will build into a significant coherent package the student can look back on with pride. Indeed, learning normally involves recreating something which already exists. Expecting mathematical apprentices to write software which already exists is not a bogus task, quite the reverse. Making a toy tool builds an appreciation and respect for the real thing.

There are technology specific issues, such as efficiency and accuracy, which contemporary students need to understand. The apprenticeship model naturally addresses these.

Computer languages come and go. Some languages are better than others. The apprentice will need to appreciate this problem, and this proposal is agnostic to language. This is a realistic stance. See, for example *Project Euler*, <https://projecteuler.net/> where people solve mathematical coding problems in whatever language they choose. Once they have the correct answer they earn the right to engage in the online discussion and see other people's solutions in other programming languages. Unlocking, in the sense of Project Euler, gets you access to the discussions and for the mathematical apprentice reaching a particular standard will unlock the in-built professional packages.

A case might be made that discussions of the theoretical limits of calculus should be placed in the real analysis course, rather than in the calculus course. However, real analysis courses almost exclusively provide expositions of the theory of integration (e.g. Riemann or Lebesgue integration), and so do not discuss computational aspects such as the algorithm for obtaining anti-derivatives in terms of elementary functions.

Calculus remains a largely paper-based subject, dominated by recipes and traditional methods. Students are still expected to learn calculus methods which are outdated, and upon which no current CAS are based. The mathematical theory for the limitations of symbolic integration in terms of elementary functions is simply not taught and is hardly mentioned. The mathematical apprenticeship offers a practical suggestion for addressing these disconnects.

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# Quantitatively Based Summation: A framework for student conception of definite integrals

Courtney Simmons<sup>1</sup> and Michael Oehrtman<sup>2</sup>

<sup>1</sup>Oklahoma State University, United States; [courtney.simmons@okstate.edu](mailto:courtney.simmons@okstate.edu)

<sup>2</sup>Oklahoma State University, United States; [michael.oehrtman@okstate.com](mailto:michael.oehrtman@okstate.com)

In 2013 Jones applied Sherins' (2001) symbolic forms to students' interpretations of the definite integral, classifying their reasoning in terms of area under a curve, function matching, and adding up pieces. Jones (2015) later refined his description of the adding up pieces symbolic form as a multiplicatively-based summation (MBS) conception that was "highly productive" in engaging students in both the mathematical structure of the definite integral as well as in modeling for physics-based contexts. This Riemann sum interpretation of the definite integral  $\int_a^b f(x) dx$  focuses on adding up many terms derived from a multiplicative relationship between a (possibly) varying integrand  $f(x)$  and a small  $\Delta x$ , a quantitative conception that we call a *local Riemann product*. The same quantitative structure may also be conceived with an infinitesimal differential,  $f(x) \cdot dx$ . The products in this conception are derived by partitioning the whole using a parallel multiplicative relationship that what we call a *basic model* which is valid for *constant* quantities. Such decompositions to the Riemann sum structure have been stressed by math, engineering, and physics education researchers as essential for STEM student success, including mathematics majors (Jones, 2015; Meredith & Marrongelle, 2008; Sealey, 2014). In 2015, Oehrtman showed that an MBS interpretation was insufficient for students constructing definite integrals from *basic models* that are not a product or that do not partition into a *local Riemann product*. Our study seeks to further explore ways in which students conceptualize definite integrals in these cases and identify key aspects of their reasoning which lead to productive results. Specifically, we pose the question: What interpretations of definite integrals and basic models are productive for students as they progress through increasingly complex definite integral modeling tasks?

Because our study seeks to understand students' progress beyond an integrand $\times$ differential quantitative relationship, we recruited students who were likely to have already developed some MBS conception. Six participants were chosen to work in pairs from a second semester calculus course in which the instructor utilized a calculus curriculum emphasizing quantitative relationships in the Riemann sum (Oehrtman, 2016). Each group was first asked to discuss their general interpretation of a definite integral, followed by a series of tasks chosen to reveal their reasoning while modeling with definite integrals progressing from simple rate $\times$ time contexts to situations that obscure the product structure of the differential form such as electrostatic force from an inverse square law. We analyzed the problem-solving and modeling activity of the students through Dewey's theory of inquiry (Dewey, 1938) and Thompson's (2011) theory of quantitative reasoning.

The result of our open & axial coding (Strauss & Corbin, 2015) is a framework that characterizes how students work through definite integral tasks which we call a Quantitatively Based Summation (QBS) framework for the definite integral. Generalizing from the MBS conception, the interpretation of the definite integral in a QBS focuses on the rich quantitative reasoning within and among what

we call the *basic*, *local*, and *global models*. The *basic model* represents the quantitative relationship which would apply to the situation if the quantities involved were constant values, the *local model* is a localized version of the basic model applied to a sub region of the original situation (typically within a partition), and the *global model* is derived from an accumulation process applied to the local model, whose underlying quantitative reasoning is encoded in the differential form.

We briefly illustrate the components and relationships of the QBS framework with a sample of work from students, Brian and Caleb (see Figure 1). This pair was able to successfully complete and justify all four tasks of the sequence.

When pollen from red cedar trees is released from their cones it travels through the air. Pollen from a mature tree settles on the ground with an estimated density of  $\delta(r) = 37/(10+r)$  g/m<sup>2</sup> a distance  $r$  meters from the tree.

Write an integral that gives the mass of pollen distributed within 100 meters from a mature tree.

Handwritten work showing the integral:  $\int_0^{100} \frac{37}{10+r} 2\pi r dr$ . Labels: density, Area. Diagram: A circle with radius  $r$  and a thin annulus of width  $dr$  and circumference  $2\pi r$ . Below:  $A = 2\pi r dr \text{ m}^2$ ,  $\delta \cdot A$ ,  $\frac{37}{10} \cdot \text{m}^2 = g$ .

**Figure 1: Mass of pollen task along with Brian and Caleb’s board work**

Brian and Caleb articulated that they think of a definite integral similar to a summation, and that in order to obtain the total mass they must add together smaller masses. This conception was tied to their general interpretation of a definite integral (*global model*) as accumulating the result of a quantitative operation when one or more of the quantities of a *basic model* varies. While discussing the relationship between the integrand and differential, Brian explained,

Because the density of pollen is not uniformly distributed you have to find, you have to find what the density of pollen is at every different  $r$ . And it’s really, it’s similar where two  $r$ ’s are really similar. So your  $dr$  is your change in that  $r$ , and within this range the density is gonna be really similar but if you just say multi, put that plugged in to 100 you’d be assuming uniform density over the whole circle, which would be incorrect.

Expressing that a radius of 100 m in his *basic model*,  $[\text{mass}] = [\text{density}] \cdot [\text{area}]$ , would presume uniform density, Brian justified a need to partition the situation and develop a *local model*. Brian reflexively recognized the density is roughly constant within concentric thin annuli, that the relevant quantities are determined by  $r$ , and the problem is geometrically and analytically partitioned by small increments  $dr$ . As a result, using the *basic model* as the structure for the overall quantitative relationship desired, Brian developed a parallel *local model*,

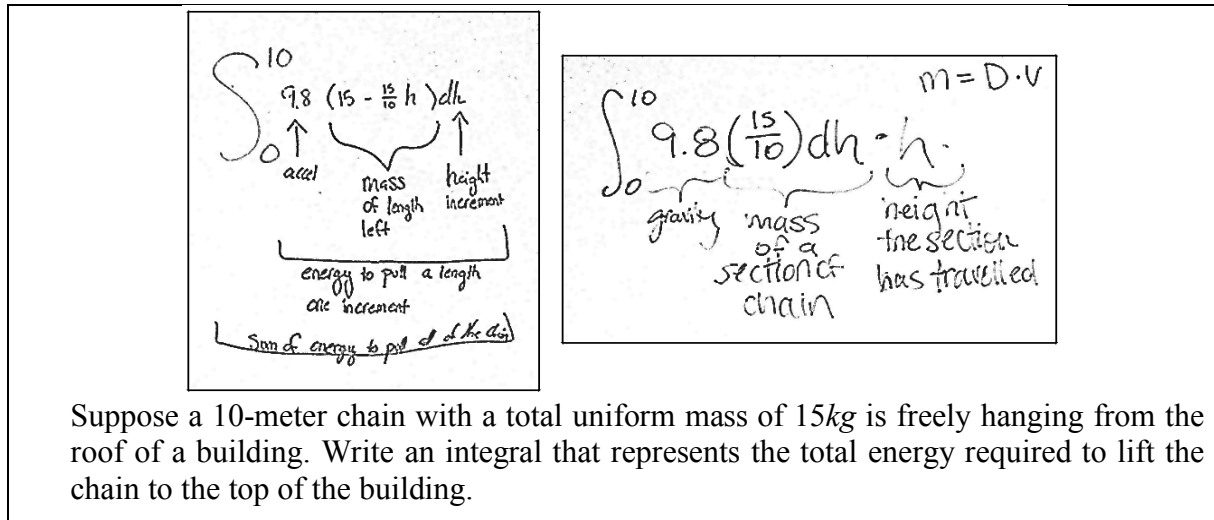
$$[\text{mass in ring of radius } r] = [\text{density at } r] \cdot [\text{area of ring of radius } r].$$

Because Brian and Caleb observed that density is nearly uniform on these regions they felt that this partition generated accurate estimates, which accumulated to a *global model* (expressed as an integral). When asked to justify the  $2\pi r$  in the [area of ring of radius  $r$ ] component of their *local model* Brian clarified,

You have the circumference, because that’s the distance around the circle, which when you cut the circle and spread it out and make it into a rectangle that would be the length of the rectangle.

But in order to get an area you have to multiply a rectangle length times its width. And your width would be your  $dr$  which is your change in  $r$  which is a very small number where the density is nearly uniform.

For Brian and Caleb the differential,  $dr$ , was a quantitative component of the area measurement within their *local model*, not a stand-alone small change in radius; a quantitative relationship which represents a distinction from the *local Riemann product* structure described in the literature.



**Figure 2: Two quantitative interpretations of the energy task: Matt (left), Julia (right)**

It is important to note that the quantitative reasoning needed within the *basic*, *local*, and *global models*, including the relationships between the models, is often non-trivial. There is also not always a canonical way to develop a *local model* from a *global model*. For example, another pair of students, Matt and Julia, was presented with a task involving the energy required to lift a chain to the top of a building. While using the same *basic model*,  $[\text{energy}] = [\text{acc\_of\_gravity}] \cdot [\text{mass}] \cdot [\text{height}]$ , they individually conceived of two different conceptual partitions of the energy in this situation; these interpretations resulted in different appropriate *local models* (see Figure 2). As Matt was constructing his *local model* he anticipated the integral summing the energy required to pull up the remaining chain over small increments. This interpretation resulted in Matt developing a *local model* that was quantitatively a *local Riemann product*, consistent with the integrand  $\times$  differential quantitative relationship described by Jones (2013, 2015). Julia, on the other hand, conceived of the integral summing the energies required to lift each small section of chain. This required that she partition the mass along small portions of the chain and resulted in the differential being an intrinsic component of a *local model* that was not a *local Riemann product*, as the differential was quantitatively conceptualized as part of the mass. To reorganize, or separate, the differential from the mass component of the *local model* into an  $f(x) \cdot dx$  multiplicative structure would result in the loss of quantitative meaning for the differential.

As illustrated in these examples, when creating a definite integral to model a quantity in a situation that was not a routine exercise for the students, they developed elements of their *basic*, *local*, and *global models* in highly nonlinear ways. In some tasks, students partitioned non-product *basic models*, such as an inverse square law for electrostatic force, into *local models* with a parallel quantitative

structure. In others, students constructed completely different *local models* based on their *global model* partitioning. In some instances, students thought simply of a *basic model* as a single quantity  $A$  with a differential  $dA$  as the corresponding *local model*. Although it is theoretically possible to reinterpret such *local models* as *local Riemann products*, for example  $[\text{mass}] = [\text{concentric density}] \cdot [\text{annulus width}]$  in the pollen task, no student was observed to do so. Correct and incorrect steps toward a solution were also heavily influenced by the students' quantitative reasoning within a symbolic form for the definite integral.

This study and resulting QBS framework highlight the importance of quantitative reasoning in the development of *local and global models* in definite integral modeling tasks. This suggests specific attention should be devoted to the quantitative relationships within 1) the differential form as a *local model*, including those that are not *local Riemann products*, 2) the symbolic form of the definite integral, and 3) the interplay between *basic, local, and global models* in the development of the differential form.

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# Examining unresolved difficulties with school algebra in calculus

Sepideh Stewart<sup>1</sup> and Stacy Reeder<sup>2</sup>

<sup>1</sup>University of Oklahoma, United States of America; [sepidehstewart@ou.edu](mailto:sepidehstewart@ou.edu)

<sup>2</sup>University of Oklahoma, United States of America; [reeder@ou.edu](mailto:reeder@ou.edu)

## Introduction

Most university mathematics instructors are aware of the shortcomings significant numbers of students have with algebra and how they impact students' success in university mathematics courses. McGowen (2017) notes that many who teach mathematics at the university level dismiss this issue as “not my problem” or “just algebra” and do not do anything to keep this from being the downfall for many students. This problem may not be well understood. Much research has focused on understanding students' difficulties with school algebra (e.g. Booth, Barbieri, Eyer, & Pare-Blagoev, 2014; Stacey, Chick, & Kendal, 2004), and on students' difficulties with Calculus (e.g. Bressoud, Mesa, & Rasmussen, 2015; Tallman, et al., 2015), but research on students' challenges with school algebra specifically in calculus courses is scarce. Reeder (2017) and McGowen (2017) cite a procedural orientation in school mathematics as the culprit for leaving students with a superficial and fragmented understanding of algebra that creates challenges for them in university level mathematics. Stewart (2017, p. vii) points out, “in college as the complexity of mathematical ideas increases rapidly, the unresolved high school algebra problems mount up progressively and continue to create further distress.” The purpose of this study is to examine students' thinking as they encounter algebraic problems within a calculus context to shed light on the origin of these difficulties.

## Theoretical framework

Our framework is based on Skemp's (1979) model of intelligence. We drew on a coherent segment of his model and utilized examples applicable to our study to analyze students' mathematical thinking and actions (Reeder et al., 2019). Skemp's model claimed that most human activities are goal oriented. To explain how humans organize their actions, he used the metaphor of a *director system*, and defined the idea of *knowing that*, as possessing an appropriate *schema*. In his view a “schema is a highly abstract concept” (p. 167). He defined “a *path* as a sequence of states and a *plan* consists of (i) a path from a present state to a goal state; (ii) a way of applying the energies available to the operators in such a way as to take the operand along the path” (p. 168). He further described “the connection between *knowing how* and *being able* to is the connection between having arrived at a plan and putting it into action” (p. 184). In his view, a “prerequisite for the production of these plans is understanding: the realization of present state and goal state within an appropriate existing schema” (p. 170). Based on some preliminary work with this model, our working research questions are: How do students respond to algebra processes in the context of calculus problems? What are their plans and what paths do they take to reach their goal state?

## Research activities

Combining our resources and expertise as mathematics educators working in the mathematics department (Stewart) and at the college of education (Reeder), we began our work by showing

examples of consequences of algebra shortcomings in college courses (Stewart & Reeder, 2017a) and launching a research study focused on identifying and cataloguing the most common algebra errors made by students in introductory mathematics courses. Our first study involved assessing exam questions from nearly 600 college algebra, business calculus, and calculus I students' exam scripts (Stewart & Reeder, 2017b) and establishing the common algebra mistakes (sign errors, errors with radicals, etc.). We then hypothesized that students might be able to successfully deal with algebra if it were free from the context of calculus. Thus, we developed an instrument (see Table 1) that included three calculus problems and some algebra problems that paralleled the algebra required to successfully complete the calculus problems. This allowed us to determine if students struggled with algebra when it was within the context of a calculus problem but not when it was in a form similar to the way they would have experienced in high school. At the end of a 16-week semester, 84 calculus students were included in this study. After coding (see Table 2) and analyzing the data, our findings were somewhat inconclusive; students made errors in the calculus problems and with the algebra problems and the errors were not consistent. When asked which was more challenging, algebra, calculus, or both, 57% indicated algebra, 31% indicated calculus, while 12% indicated both. Their comments overwhelmingly expressed their frustration and anxiety. One student wrote: "I felt as if I hadn't learned anything or retained anything...I want to be better at math, but I don't know how."

**Table 1: The Calculus and Algebra questions**

Calculus questions	Algebra questions
1. Implicitly differentiate. $\sqrt{xy} = 1 + x^2y$ 2. Find the critical numbers of the function $f(t) = t\sqrt{4-t^2}$ 3. Evaluate the limit. $\lim_{t \rightarrow 0} \frac{\sqrt{1+t} - \sqrt{1-t}}{t}$	1. Solve for y. $5 + xy = 10 + x^2y$ 2. Solve for y. $\frac{1}{2\sqrt{5x}}(5 + xy) = 10x + x^2y$ 3. Solve for t. $\frac{1}{\sqrt{t+1}} - \frac{1}{t} = 0$ 4. Solve for y. $\frac{2y^2}{2\sqrt{y^2-9}} + \sqrt{y^2-9} = 0$
My main problem with the test was: Algebra <input type="checkbox"/> Calculus <input type="checkbox"/>	
Please write a comment relevant to your experience in taking this test.	

**Table 2: Potential errors for Calculus and Algebra contexts**

Possible Calculus Errors	Possible Algebra Errors	Other Possible Error
1. Power Rule 2. Product/Quotient rule 3. Chain Rule 4. Process of Implicit 5. Interpret Critical Numbers (set =0) 6. Undefined points are Critical 7. Taking the limit	8. Convert radical to exponent 9. Exponent Operations 10. Balance Points 11. Distributive Property 12. Combining Like Terms 13. Cancelling 14. Factoring 15. Simplifying nested fractions 16. Sign error 17. Operations with radicals 18. Finding Common Denominators 19. Recognizing undefined values 20. Conjugating Rational Fractions 21. Quadratic Functions 22. Operations with Fractions	23. Blank 24. Incomplete Calculus 25. Incomplete Algebra 26. Computation 27. Avoiding Algebra 28. Avoid Calculus 29. Miscellaneous 30. Isolating Variables

Based on our research we have established two common types of calculus problems with corresponding algebra occurrences in those problems. These are presented as problems wherein the



calculus precedes the algebra (Type 1, LHS) and wherein the algebra precedes the calculus (Type 2, RHS) (see Table 3). Analysis of both types of Calculus I problems reveals that in Type 1 calculus problems, many students can take the first derivative, but cannot carry out the many steps of algebra to complete the problem (Stewart & Reeder, 2017). Likewise, analysis of Type 2 calculus problems reveals that many students either try to avoid the algebra in the first steps altogether, or have difficulty with the algebra that often involves rationalizing the denominator, or factoring, which results in incorrect answers (Reeder, et al., 2019).

**Table 3: Type 1 (LHS) and Type 2 (RHS) examples.**

<p><b>1. Find the absolute maximum and absolute minimum values of <math>f(x) = x\sqrt{x - x^2}</math></b></p> <p>Differentiating yields:</p> $= x \frac{1 - 2x}{2\sqrt{x - x^2}} + \sqrt{x - x^2}$ <p>Combine into a single fraction</p> $= \frac{(x - 2x^2) + (2x - 2x^2)}{2\sqrt{x - x^2}} = \frac{3x - 4x^2}{2\sqrt{x - x^2}}$ <p>Set <math>f'(x) = 0</math></p> $\frac{3x - 4x^2}{2\sqrt{x - x^2}} = 0$ $\Rightarrow 3x - 4x^2 = 0 \Rightarrow x(3 - 4x) \Rightarrow x = 0 \text{ or } x = \frac{4}{3}$ <p>Test boundaries and critical points</p> $f(0) = f(1) = 0 \text{ absolute minimum}$ $f\left(\frac{3}{4}\right) = \frac{3}{4} \sqrt{\frac{3}{4} - \left(\frac{3}{4}\right)^2} = \frac{3}{4} \sqrt{\frac{3}{16}} = \frac{3\sqrt{3}}{16}$	<p><b>Evaluate <math>\lim_{x \rightarrow 3} \frac{x^2 - 9}{\sqrt{x} - 3}</math></b></p> <p>Rationalize the denominator:</p> $\lim_{x \rightarrow 3} \frac{x^2 - 9}{\sqrt{x} - 3} \cdot \frac{\sqrt{x} + 3}{\sqrt{x} + 3} = \lim_{x \rightarrow 3} \frac{(x^2 - 9)\sqrt{x} + 3}{x - 3}$ <p>Factor the numerator:</p> $\lim_{x \rightarrow 3} \frac{(x + 3)(x - 3)\sqrt{x} + 3}{x - 3}$ <p>Cancel the common factor:</p> $\lim_{x \rightarrow 3} \frac{(x + 3)\cancel{(x - 3)}\sqrt{x} + 3}{\cancel{x - 3}} =$ $\lim_{x \rightarrow 3} (x + 3)\sqrt{x} - 3$ <p>Evaluate at <math>x = 3</math></p> $= (3 + 3)\sqrt{3} - 3 = 6(0) = 0$
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## Current work

In a case study in Fall 2017, at the end of the semester prior to final exams, we administered a modified version of our calculus and algebra tests to a group of 33 calculus I students. For this paper we analyzed one set of calculus/algebra questions related to limits (see Table 4). We found that only 10 students performed the correct procedure. These students also performed well doing the algebra questions. Among the 10 students, seven chose calculus as their main problem, two said algebra and one said both algebra and calculus. Naturally, we expected to hear that students had difficulties with calculus, and not with algebra. Among those who were unsuccessful, some claimed

**Table 4: Calculus students' responses to the limit question**

<p>Evaluate <math>\lim_{x \rightarrow 3} \frac{x^2 - 9}{\sqrt{x} - 3}</math></p> <p><math>\lim_{x \rightarrow 3} = \text{DNE}</math></p> <p><math>\lim_{x \rightarrow 3} = \text{DNE}</math></p>	$\left(\sqrt{x-3} + 2x\right) - \left(x^2 - 9 + \frac{1}{2\sqrt{x-3}}\right)$ $\frac{2x\sqrt{x-3} - \frac{x^2-9}{2\sqrt{x-3}}}{(x-3)} = \frac{2x\sqrt{x-3}(2\sqrt{x-3}) - \frac{x^2-9}{2\sqrt{x-3}}}{(x-3)} = \frac{4x(x-3) - \frac{x^2-9}{2\sqrt{x-3}}}{(x-3)}$
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that the limit does not exist, others got involved in doing endless algebra with no way out, and some students left the question blank. To take a closer look at students' reasoning, we interviewed several

calculus students and using Skemp's model (1979) examined their thought processes solving the same limit problem. The result showed that students struggled to resolve their algebra issues in the context of this problem (Reeder, et al., 2019). Currently, we are refining our instruments and hoping to theoretically explain why these problems are persistent and find ways of incorporating interventions to improve instruction of calculus. We are concentrating on the limit problems, as they are fundamental to calculus and often create enormous calculation and conceptual difficulties.

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# Construction of the mathematical meaning of the function-derivative relationship using dynamic digital artifacts

Osama Swidan

Ben Gurion University of the Negev, Israel; [osamasw@gmail.com](mailto:osamasw@gmail.com)

## Introduction

Studies that investigated the learning of the derivative symbolically and graphically have found that students tend to refer to the symbolic register when they learn about derivative (Yoon and Thomas, 2015). Despite this tendency, several scholars have argued that understanding the mathematical relationship of the function and its derivative graphically may lead to conceptual understanding of the concept of derivative (Haciomeroglu et al., 2010; Ariza et al. 2015). To improve students' conceptions of the derivative, mathematics educators have been reflecting on their students' endeavors, with emphasis on conceptual developments in a technological environment (Berry and Nyman, 2003; Ubuz, 2007). Most of these studies, however, have investigated students' graphic understanding of the derivative in different tasks, and only a few have looked into how students construct mathematical meaning of the function-derivative relationship as they interpret the linked-dynamic function and derivative graphs. And relatively few studies have explored the role of dynamic digital artifacts in prompting the construction of the mathematical meaning of the function-derivative relationship.

This paper considers the use of dynamic digital tools for developing the understanding of the function-derivative relationship. In doing so, it aims also to reveal the general potentials of dynamic and linked representations in learning fundamental calculus concepts, and to provide some hints regarding the theoretical principles behind them. The main objective of this study is to characterize how students endow with meaning the function-derivative relationship when learning with dynamic digital artifacts. To this end, we used the semiotic mediation theory (SMT) (Bartolini Bussi and Mariotti, 2008) to identify the evolution of personal meanings of the function-derivative relationship, and how this meaning evolves into mathematical meaning.

## Theoretical framework

The SMT (Bartolini Bussi and Mariotti 2008), which guided this study, claims that the relation between artifact and knowledge is expressed by culturally determined signs and that the relation between the artifact and the learners in the course of accomplishing a specific task is expressed by signs such as speech, gestures, symbols, and tools. Bartolini Bussi and Mariotti (2008) suggested a model of a learning process that takes advantage of the potential of artifacts. The model aims to describe how meanings related to the use of a certain artifact can evolve into meanings recognizable as mathematical. The SMT assumes that social interaction and semiotic processes play a key role in learning, particularly in situation in which learners are encouraged to use the artifact in order to solve a given task. The SMT considers learning to be an alignment between the personal meanings arising from the use of a certain artifact for the accomplishment of a task and the mathematical meanings that are deployed in the artifact. The SMT describes the relations between personal meanings and mathematical meaning as a double semiotic relationship. On one hand, we concentrate on the use of


the artifact for accomplishing a task, recognizing the construction of knowledge within the solution of the task. On the other hand, we analyze the use of the artifact, distinguishing between the constructed personal meanings derived by individuals from their use of the artifact in accomplishing the task and meanings that an expert recognizes as mathematical when observing the students' use of the artifact in order to complete the task.

Various kinds of signs are produced in the practical activity with the artifact. The SMT distinguishes between three kinds of signs: artifact, pivot, and mathematical. Artifact signs refer to the artifact and its use. These signs may evolve into mathematical signs that refer to the mathematics context. The mathematical signs are related to the mathematical meanings shared in the institution to which the classroom belongs. Through a complex process of evolution of the artifact sign into a mathematical sign, other types of signs, called pivot signs, play a crucial role. The characteristic of these signs is their shared polysemy, that is, they may refer to the activity with the artifact as well as to natural language and to the mathematical domain.

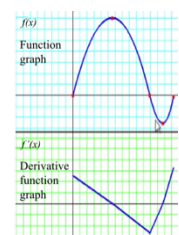
## Method

### The Calculus Derivative Sketcher and its Features

The dynamic digital artifact used in this study is the Calculus Derivative Sketcher (CDS) (Shternberg et al. 2004), in which two dynamically linked graphs are constructed according to the function-derivative relationship (Figs. 1).

The basic features of the CDS comprise *construction*, which allows one to generate a graph of a function on the upper Cartesian system of the CDS using the icon buttons; and *dragging*, which allows one to manipulate (move, stretch, etc.) the graphs. The design of the CDS determines how the construction and dragging functions are actually performed. The graphic user interface of the CDS contains seven icons  that are used to cover the different situations of a one-variable function graphs (i.e., constant function, decreasing and increasing linear functions, decreasing and concave-up functions, decreasing and concave-down functions, etc.)

(Schwartz and Yerushalmy, 1995). Users can sketch a graph of a function on the upper Cartesian system by using one or more of the icons. In response, the CDS displays a function graph consisting of segments whose shapes resemble the respective icons that were chosen. In all cases, the CDS generates derivative function graph in the lower Cartesian system.



**Fig. 1 CDS interface**

### Participants, procedure, data collection and analysis


To analyze in-depth the evolution of the personal meanings toward mathematical meaning, this study focused on a case of Manhal, a 15-year-old student who volunteered to participate in after-school meetings. At the time of case study, the students had already learned the concepts of linear function, including the concept of the slope, and of the quadratic function, but not that of derivative. The experiment was conducted at a computer lab at the student's school. The interviewer briefly introduced the student to the artifact and showed how to work with it. To identify the processes of evolution of personal meanings of the derivative function toward its mathematical meaning, the interviewer asked the student to graphically explore and explain out loud the possible connection

between the two graphs on the screen. To collect the data, the learning experiment was recorded entirely with two video cameras. One camera was located behind the student to capture his gestures of pointing to the computer screen. The second camera was set in front of the student to record him and the computer screen. The entire videos were transcribed. The findings presented below are the result of three rounds of data analysis. The first round consisted of repeatedly reading the transcripts to identify the mathematical elements involved in finding the possible connections between the graphs. Next, we looked for the correlations between the two linked graphs identified by the student. In the third round of analysis, we identified how the personal meanings evolved into mathematical meanings by distinguishing between artifact signs, pivot signs, and mathematical signs. In addition, actions with the artifacts and student's gestures were also analyzed.

## Preliminary results

In total, five mathematical elements were found: (a) recognizing the relationship between the two graphs; (b) recognizing the tangent slope of a curve; (c) recognizing the extremum point in the function graph; (d) recognizing the meaning of the inflection point in the function graph; (e) recognizing the meaning of the concavity of the function graph. Due to space limitation I will focus here just in element (b). During the conference I will elaborate on all the elements.

During the learning experiment, the interviewer encouraged Manhal to create a new function and to examine a variety of graphs.

Manhal: [I] Now I choose a **curved graph** [presses the fourth icon  (Figure 6a)]. What is **slope**?

[II] No, what is the characterizations of the **slope** here? [III] I see that the **slope** was a **specific value**, and it begins **decreasing**. Why is it decreasing? If I move on the graph like this [gesturing with his left hand (Figure 7)], as long as I go, the **slope decreases**. If I add the first graph here (Figure 6b).

Interviewer: Okay!

Manhal: You see, **its slope is constant**. Here the **slope** is varying because the **function is curved**. Here, too, the **slope descends**, it starts at two and begins to **descend**.

In utterance [I] Manhal created an artifact sign using the fourth icon. His words “curved graph” suggested that Manhal had endowed the artifact sign with mathematical meaning—a curved graph. His question “what is slope?” suggests that he was referring to the same mathematical sign “slope” he used previously. Changing his question from “what is slope?” to “what is the characterization of a slope?” suggested that Manhal changed his strategy of making sense of the function graph. To come up with a conjecture regarding the characterization of the function slope, Manhal used three mathematical signs: “slope,” “specific value,” and

“decreasing.” He used the last two signs to determine the slope behavior. His words “I see that” suggest that Manhal was perceiving the slope variation with his eyes. To answer the questions he raised, Manhal performed a “*surfing gesture*” by his left hand, showing the way a tangent line of the function graph may behave (Figure 7). Doing so, Manhal used the word “move on” as a pivot sign to exemplify the tangent movement on the graph. The use of the pivot sign and the gestures afford

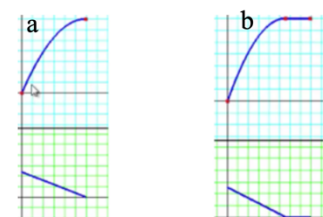


Figure 2. (a) Concave-down function graph (b) a constant function graph added to the concave-down function graph

Manhal to correlate between the movement on the function graph and various values of the slope of function (“as long as I go the slope decreases”). To verify his conclusions, Manhal added the first segment to the curved graph, thus obtaining an artifact sign that consists of two segments. Manhal endowed the artifact sign with mathematical meanings as a function in which the slope is constant. He also made a comparison between the two segments based on their shapes. Namely, Manhal endowed the artifact signs with mathematical meaning—as a function—and characterized them according to their typicity: one is constant whereas the other is curved. Manhal shifted his attention from the upper system to consider the lower Cartesian system. His words “the slope descends, it starts at two and begins to descend” suggest that Manhal was using the slope to name the Cartesian system.



**Fig. 3. Manhal illustrating the change of the tangent slope of a concave-down function graph.**

### Final remarks

This study sheds light on the process of artifact signs evolving into mathematical signs by using pivot signs. In my talk I will discuss the evolution of the word “slope” and its role in constructing the meaning of the function-derivative relationship. This study stressed the importance of the word “slope,” which plays a role not only in connecting the graphic and symbolic registers in the definition of the limit as a process (Lobato et al., 2012; Park, 2015), but also in graphically constructing the meaning of the function-derivative relationship. The word “slope” assumed several meanings in the course of the learning experiment. At the beginning of the experiment, it was used as a numerical value for a straight line inclination “the slope of the line is 1;” later the word “slope” was used as a name of the lower Cartesian system (“this system is the slope”); as a dynamic object which has certain properties (increase/decrease, positive/negative), by which the student described the shape of the graph; and, toward the end of the learning experiment, to describe the shape of the graph segments. This evolution of the meanings of the word “slope” indicates that Manhal transitioned from the point-specific view to the interval view (Park, 2015), and ultimately constructed the meaning of the derivative function. This transition will discuss in depth in my talk.

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# Linear stability or graphical analysis? Routines and visual mediation in students' responses to a stability of dynamical systems exam task

Athina Thoma and Elena Nardi

University of East Anglia, United Kingdom; [a.thoma@uea.ac.uk](mailto:a.thoma@uea.ac.uk), [e.nardi@uea.ac.uk](mailto:e.nardi@uea.ac.uk)

## Introduction

Dynamical systems are a crucial topic of differential equations, taught in Mathematics as well as in other, e.g. Engineering, departments. In the UK, where the study we report from was conducted, differential equations are usually introduced in the second year of study after first year courses in Calculus. Despite extensive research in elementary Calculus, at large, differential equations, and specifically dynamical systems, are not widely researched topics. Previous work on dynamical systems highlighted students' use of different procedures, analytical, graphical and numerical methods and the importance of graphical representations (Rasmussen, 2001). Rasmussen comments on the methods needed for the solution of differential equations and notes that:

“With graphical (or qualitative) methods, one obtains overall information about solutions by viewing solutions to differential equations geometrically and by analyzing the differential equation itself” (Rasmussen, 2001, p. 56).

By analysing the work of six undergraduate students, Rasmussen offers a framework aimed at investigating students' understanding of the graphical and numerical methods used to solve differential equations. His results show that, while students use various methods, they do not make connections between these. Rasmussen notes the scarcity of studies in this area, as does Artigue (2016), referring to her own work done as early as the 1980s. Artigue also discusses the limited experiences that undergraduate students are offered in terms of differential equations with the main focus being on procedures of solving these equations (*ibid.*, p. 13). Both studies stress that the focus of the students' experience in solving differential equations is on the procedures and that this experience is excessively compartmentalised.

The data analysis we present here – in which we take a discursive perspective, Sfard's (2008) theory of commognition – takes this discussion further by investigating the *how* (procedure) and the *when* (applicability and closing conditions) of a routine (*ibid.*, pp. 208-209) as evident in students' written responses to an exam task that asked them to examine the stability of equilibrium points of two dynamical system via linear stability analysis or graphical analysis. We note that the commognitive perspective views mathematics as a discourse characterised by *word use* (e.g. dynamical systems, equilibrium points), *visual mediators* (e.g. sketches of functions), *endorsed narratives* (e.g. definition of equilibrium point) and *routines* (e.g. examining the stability of equilibrium points). The data and analysis we report here are part of a larger study, conducted in the UK, in a well-regarded mathematics department. The study investigates students' engagement with university mathematical discourses in the context of final year examinations (Thoma, 2018; Thoma & Nardi, 2017; 2018).

## The task, the lecturer's aims and the students' responses

The module Differential Equations and Applied Methods was attended by ninety-seven students. The first author selected thirty-four student examination scripts for further analysis, to represent a variety of marks (we explain the process for another module in Thoma & Nardi, 2017, Figure 3 on p. 2269). Here, we focus on the responses to this task in a part of these thirty-four scripts:

- (a) Consider the one-dimensional system  $\dot{x} = f(x)$ , where  $x(t)$  is a real-valued function of time  $t \geq 0$  with  $x(0) = x_0$  given, and  $f(x)$  is a smooth real-valued function of  $x$ . Define what it means to say that  $x^*$  is an hyperbolic equilibrium point for this dynamical system. [4 marks]
- (b) Find the equilibrium points for each of the dynamical systems:  $\dot{x} = x^3 + x^2$ ,  $\dot{x} = 1 + \sin x$ , and analyse their stability. You may use linear stability analysis or graphical analysis. [6 marks]

The task initially invites students to recall and provide the definition of a hyperbolic equilibrium point for a one-dimensional system (a). Then, the students are given two different one-dimensional dynamical systems (b), and they are asked to, first, find the equilibrium points for each dynamical system and, then, analyse their stability. There is also a direct instruction regarding the procedures that they can choose from in order to analyse the stability “You may use linear stability analysis or graphical analysis”. It is this choice that this paper focuses on. The students are allowed to choose which procedure they want to use when examining the equilibrium points stability. In an interview with the exam-setting lecturer that took place after the final examination, the lecturer comments on providing the students with both of the procedures to analyse the stability as follows:

“I give them a choice, there are two methods of studying stability (...) one of them is not appropriate for all of the stability. But the idea is for them..., is to recognize that one of them...to get the full marks they have to use the other one, the graphical analysis. (...) I could have asked only for graphical analysis without linear stability but I also want to test, to see if they understand linear stability in the part which can be used.”

The lecturer's aim is dual: first, to give the students the option to use both procedures (linear stability and graphical analysis) in analysing stability; and, second, to examine whether the students can distinguish which procedure is suitable in the situation. In commognitive terms, the focus here is both on the procedure (*how*), and the applicability conditions and closing conditions (*when*) of the routine. The lecturer places value on the *when* of the routine. The dynamical systems are selected on purpose to provide an opportunity to examine the applicability of the two procedures. For one of the equilibrium points of the first dynamical system ( $x^*=0$ ), if one uses linear stability analysis, the information from the first derivative is not sufficient to decide on the stability as it gives the value zero. This is the same for the infinite equilibrium points of the second dynamical system.

Of the thirty-four students, two did not attempt the task at all. The rest mainly used graphical analysis (e.g. Figures 1 and 2). For the first dynamical system, fourteen students used graphical visual mediators to discuss the stability of  $x^* = -1$ , and twenty-nine for the stability of  $x^* = 0$ . For the second dynamical system, twenty-seven students used graphs. Specifically, for the equilibrium point  $x^* = -1$  of the first dynamical system, eighteen students used linear stability to characterise the stability by finding the value of the first derivative at that point, thirteen used graphical analysis and one provided



the graph of the function without characterising the stability of the equilibrium point. For the second equilibrium point ( $x^*=0$ ), twenty-one used graphical analysis to discuss the stability of the point. Of these, nineteen plotted  $x^3+x^2$  and two plotted the functions  $x^3$  and  $x^2$  separately; seven used graphical analysis but their sketches were not the correct function; one plotted the function but did not characterise the stability; and, three provided the characterisation without providing a sketch of the function basing their argument on linear stability arguments. Regarding the second dynamical system, twenty-four students used graphical analysis. Of these: twenty-two sketched  $1+\sin x$  as one function and two plotted the two functions separately. Further, three students used graphs in their responses but they either did not provide characterisation regarding stability of the infinite equilibrium points or provided the sketch of a different function. Finally, five students provided only comments using linear stability analysis. In the following, we discuss in detail the responses of two students regarding the stability of the first dynamical system.

Student [03] initially identifies the two equilibrium points and examines both using linear stability analysis (Figure 1); identifies (with the linear stability analysis) that the information is not sufficient to determine the stability of one of the equilibrium points ( $x^*=0$ ); and, decides to provide the graphs of  $-x^2$  and  $x^3$  in order to analyse the stability. Additionally, in the narrative accompanying the graphical realisation of the function, [03] discusses the monotonicity of the function  $x^3+x^2$  by examining what happens on the left and right hand side of the equilibrium point.

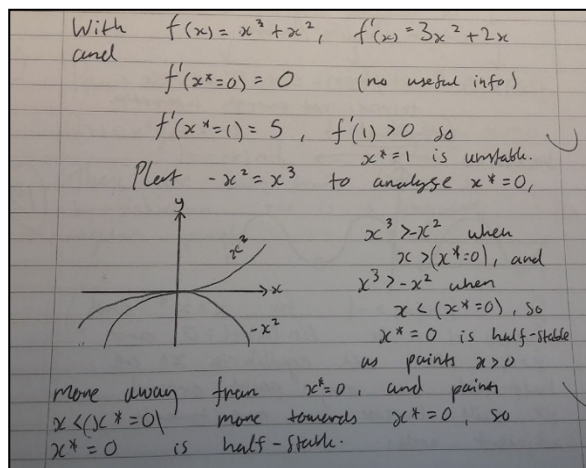


Figure 1: Student [03]' response to task (iib)

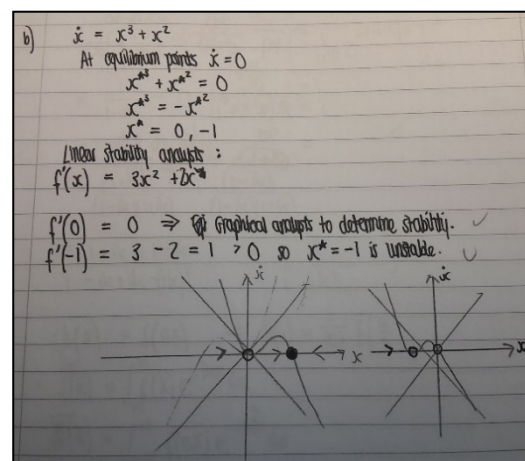


Figure 2: Student [10]'s response to task (iib)

In student [10]'s response (Figure 2), there are two graphs: both are attempts at the graphical realisation of the function  $x^3+x^2$ . However, both are crossed out (they are the graphs of  $-x^3-x^2$  and  $-x^3+x^2$  respectively, not  $x^3+x^2$ ). Although student [10] realises that the procedure to analyse the stability of the dynamical system is graphical analysis, the graphs provided are from different functions and thus not suitable to decide the stability of the equilibrium point. Both of these students decide to use linear stability analysis for  $x^* = -1$  and graphical analysis to characterise the stability of  $x^* = 0$ . The when of the routine is examined and both discuss what we classify as applicability conditions in order to use linear stability. Further, we note that, despite opting for graphical analysis, the how of the routine is different in these two responses. Student [03] provides the graphs of  $-x^2$  and  $x^3$  and student [10] attempts to provide the graph of  $x^3+x^2$ .

## Conclusion

The analysis summarised here illustrates that the students are mainly using graphical analysis in their characterisation of the equilibrium points. However, we note that there were students who used linear stability analysis even though the applicability conditions of the routine were not satisfied, namely the value of the first derivative at the equilibrium points was zero; still, the students chose to continue with the linear stability analysis. Our analysis also highlights the different procedures that are discernible in the students' responses even when they opt for graphical analysis to discuss the stability of the equilibrium points on both dynamical systems. Some sketch the actual function and others examine the parts of the function in the same coordinate system and discuss the range where one of the parts of the function is above or below the other one.

Our results highlight the importance of investigating further students' engagement with the routines the exam setter is expecting them to engage with, especially in relation to whether they take into account the applicability conditions of each of these routines and the variety of the procedures that they could use while deciding on the stability of equilibrium points in dynamical systems. Previous work discussed the compartmentalization of these procedures (Rasmussen, 2001); our work offers further discussion of whether students take on board the applicability conditions (the *when* of a routine) as well as the various procedures (*how*) of the routines. We are currently analysing analogous data focusing on examination tasks from other mathematical areas as well.

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# The ‘Signature’ of a Teaching Unit – ‘Calculus’ as the ‘Heart’ within Standard Introductory Mathematical Courses

Günter Törner

University of Duisburg-Essen, Germany; [guenter.toerner@uni-due.de](mailto:guenter.toerner@uni-due.de)

If one compares internationally the mathematical ranges of study at universities, one meets again and again Calculus as the central beginner course; presumably, this has been the case in Western European countries for more than 200 years and is due to the activities of the International Teaching Commission at the beginning of the last century. Although there are countless books on the history of Calculus, we do not yet know specific historical inventories of *Calculus courses* as a teaching unit, even though some contributions by Coray et al. (2003) and Schubring (2004) point in this direction.

Although mathematics has developed greatly in its variety, not least because of the possible use of computers and tools, and today faces infinite mathematics with a broadly developed discrete mathematics, it seems to us that, in a modification of Halmos’ (1980) style of speech, we may award the term *heart* unchallenged - as stated in the title - to Calculus courses. It is now largely forgotten that among the many US-Calculus reforms in the last century, an interesting discussion took place in 1984 in *College Mathematics Journal*, 15 (5), which was initiated by the book (Ralston & Young, 1983). These papers explore the question of whether Calculus should give way to an introductory course on discrete mathematics. Prominent mathematicians participated in this discussion, e.g. Saunders MacLane, Peter Hilton, R. W. Hamming, David Tall, Patrick W. Thompson, John Mason, etc.

This raises the question as to what makes Calculus a highlighted prominent course. In 2005, Lee Shulman introduced the term *signature pedagogy* as a differentiating feature for various scientific disciplines. Shulman's basic thesis is that nurseries are of high influence within child development. In the same way, this also applies metaphorically in our context, because the Calculus course is a first encounter with university mathematics.

We want to modify this concept and its terminology here and speak of the *signature of a course*. We ask ourselves what are the salient properties of a course constituting a ‘signature’. After that, we believe, we can better describe the central role of Calculus.

## Parameters by which a ‘signature’ of a mathematical course is constituted

Some rather superficial, quantitative traits quickly come to mind:

### i. The role of the specific domain in relation to other subfields

In mathematics, we are fortunate enough to be able to quantify the relevance of a discipline to a ranking fairly reasonably. It is well known that all internationally relevant branches of mathematics are listed by a *Mathematics Subject Classification*<sup>1</sup> Index (MSC) updated every 10 years; the currently valid MSC comes from the year 2010, a moderate revision is in discussion and planned for the year 2020. The MSC currently lists nearly 90 mathematical areas, while the area Mathematics Education

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<sup>1</sup> <https://mathscinet.ams.org/msc/msc2010.html>

appears at number 97. The term 'calculus' as such does not appear directly in this list, but we would assign 19 branches with the numbers between 26 and 49 to this area. The *didactically* relevant entries around the calculus can be found in field 97 and are subdivided into 9 subareas in category 97I xx.

## **ii. Relevance of research within the domain by counting publications**

The *Mathematical Reviews* database lists more than 1,400 entries that respond to the search term 'Calculus'. The didactic database *MathEduc* show 2,800 hits with the word 'Calculus' in the title of an article, the *zbMath* database references exhibit about 700 in the field of education for Calculus-oriented articles.

Again and again, technical and didactic thematic issues are devoted to this subject, e.g. *The Montana Enthusiast* (TME) counts 144 articles in the last four years, *ZDM Mathematics Education* has about 250 articles since 1997 and all previous *International Congresses on Mathematical Education* (ICME) organized *Topic Study Groups on Calculus*. Recently, a booklet has been created for the most recent ICME in Hamburg in 2016 (Bressoud et al., 2016) where the author was involved.

## **iii. Cross-linking degree**

Even insiders would probably not like to conclusively answer the question whether Calculus is a course that belongs to the areas of *pure mathematics* or *applied mathematics*; and most likely they would offer us an *as well as*.

## **iv. Historical dimension**

Another parameter is the historical anchoring of a discipline with archaeological traces still visible. It would be possible to determine these parameters for other topics such as linear algebra, discrete mathematics, algebra or stochastics; we refrain from doing so because we believe that the reader recognizes as proven the dominant role of the teaching unit calculus. With more than 2,600 years of experience, *Calculus* is proud to have a history which can only be surpassed by *Geometry*. *Linear Algebra* is looking back around 180 years of age whereas *Discrete Mathematics* is a mere 100 years. Corresponding messages to students should not be missing in a course.

## **v. Worldview facets of mathematics provided by calculus**

Undoubtedly, we may follow many mathematicians who believe in the *unity of mathematics* (Atiyah, 1978). On the opposite side, we have to consider that mathematics has many facets; it seems as if there were *three (different) mathematics* (Ziegler & Loos, 2014) into which each mathematical discipline can contribute at various occasions. Thus, the central question whether calculus is displaying all these facets or – to use a metaphor – these colors? Yes, indeed.

In the paper of Ziegler & Loos (2014), the authors try to characterize different world views (belief systems about mathematics) (see the first use by Grigutsch, 1996). The author constituted this term 20 years ago in the German mathematical education discussion (see Leder et al., 2002). Citing the world views (for teachers) from Ziegler and Loos and slightly extending these characteristics, we should mention:

- Functioning as a large toolbox for the everyday world of a working mathematician,
- A large field of research central in *pure* as well *applied mathematics*,

- A discipline with a long history and a *prominent part of culture* and simultaneously a *key for modern technologies*. (Hoffmann et al., 1997)

Obviously, calculus fits into each of these three categories, *thus calculus has something for all of them*. However, we would like to refer to another aspect that seems important to us, especially for beginners; this categorization can be transferred specifically to other courses.

#### **vi. The Front- and Back of Calculus**

In the sense of the sociologist Erving Goffman, it is Hersh (1991), who worked out, that mathematics has a *front* and a *back*. The purpose of a separation between a ‘front’ and ‘back’ is *not* just to keep the customers (in a restaurant) from interfering with the cooking (kitchen). Thus, separation, in reality, often serves legitimate functional reasons. But do we really like to claim a similar separation in *mathematics learning and teaching*? Textbooks of calculus show the *front*, our exercise booklets are more like the *back*. Calculus is a paradigmatic domain where the learner often has to switch between *front* and *back*. The following situation illustrates the problem. In Calculus courses, students were introduced to real numbers, continuous and finally to differentiable functions; in the later paragraphs of the courses, the classical central theorems were deduced and proved. This straightforward (linear) organization belongs to the *front*, is at first glance, optimal, common and follows an axiomatic deduction of knowledge, it is like being served in a restaurant.

However, what has happened in the *back*, in the kitchen: Continuity of functions is quite different from differentiability. The first property is just a segregation concept to exclude more nasty functions, whereas differentiability is a quality statement. Of course, it is comfortable being served in the front, but on the other hand, mathematicians are challenged to sometimes act also as a cook. Thus, instructors should often open, in specific situations, the *back* for the novice learner. It is up to the reader to exploit further examples from calculus.

For the presenter, *Calculus* is an old and a young discipline which proudly stands beside other important mathematical disciplines. Thus, calculus is indispensable and formative in the education of young mathematicians.

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# **What to do when there is no formula? Navigating between less and more familiar routines for determining velocity in a calculus task for engineering students**

Olov Viirman and Irina Pettersson

University of Gävle, Sweden; [olov.viirman@hig.se](mailto:olov.viirman@hig.se), [irina.pettersson@hig.se](mailto:irina.pettersson@hig.se)

In this paper, we describe and analyze the “Svensson’s vacation” problem, designed for a course in Single-Variable Calculus aimed at first-year students in a three-year engineering program. Although it is not formally required, most students enrolling in the course will have taken courses in introductory and linear algebra. The problem was developed for a small-scale intervention aimed at introducing elements of inquiry-oriented teaching. In some of the exercise sessions, the students were asked to work collaboratively on one or two larger problems designed to support and deepen their understanding of the central topics of the course. We wanted problems that include elements of modelling and numerical methods, which is not a part of the curriculum in any of the obligatory mathematics courses in the three-year engineering program, but which is very useful for future engineers (Alpers et al., 2013). Still, we wanted problems to be doable with limited resources, without having to rely on access to computer labs or specialized software, for instance. In the analysis of the task, we employ a discursive approach, the theory of commognition (Sfard, 2008), building on the characterization of mathematical discourse through word use, visual mediators, endorsed narratives and routines (Sfard, 2008, p. 133). We describe the setting of the problem first.

## **The “Svenssons’ vacation” problem**

The problem is intended for the session on derivatives, with a follow-up as part of the session on integrals. The problem consists of two parts. The first is a warm-up task introducing numerical differentiation and the notions of forward and central differences. Students are provided with a table of values of a function, and are asked to approximate derivatives at a number of points using the table and the difference quotients. Using the analytical expression for the derivative as a comparison, they are then asked to estimate the error in the approximations. The main problem then reads as follows:

The Svenssons were going on vacation to Thailand. They packed their bags, got into their car and drove off for the airport. After a while, they started worrying that they might have forgotten their passports. They drove off at the next exit, and started rummaging through their bags. Sure enough, the passports were nowhere to be found. They turned around and drove back home. By now, they were in a bit of a hurry. When you’re under stress, finding what you’re looking for takes longer, but it turned out alright in the end, and the Svenssons got to the airport on time.

Attached you see a graph (Figure 1) showing the time in minutes and the position of the Svenssons’ car at the corresponding time, measured in kilometers from their house. Please answer the following questions:

What was the velocity of the car at times  $t=3$ ; 10; 22; 55 minutes?

What was the average velocity of the car in the time intervals  $[10,15]$  and  $[20,25]$  minutes?



Sketch a graph showing the velocity of the car as a function of time. Does the graph look reasonable? Does it agree with the story of the Svenssons?

During the session on definite integrals, the bulk of which was spent on numerical integration, we also included a follow-up to the “Svenssons’ vacation” problem. The students were given a graph of the approximation for the instantaneous velocity (Figure 2) and were asked to estimate the distance from home for Svenssons after 15 and 25 minutes respectively.

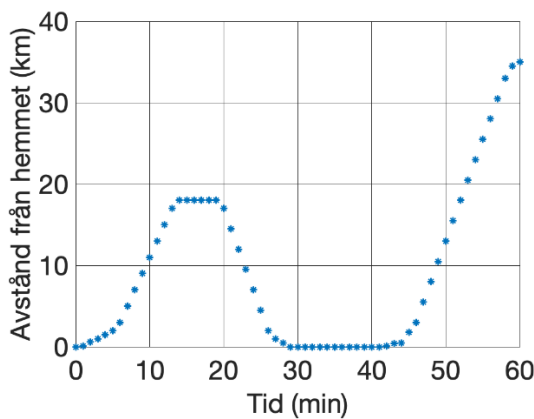


Figure 1: Position of the car at time  $t$

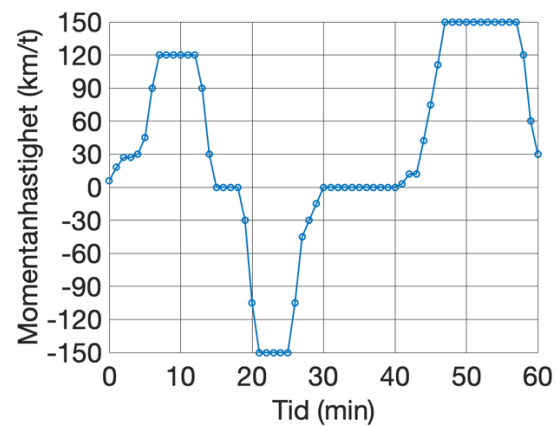


Figure 2: Velocity of the car at time  $t$

### Analysis of the problem

The problem aims at engaging students with differentiation (and integration) in a setting where functional relationships are not given by formulas, meaning that the familiar analytic differentiation routines are not readily applicable. There is also potential for students to form connections between the fundamental objects of derivative and definite integral, through exploring the relationship between rate of change (RoC) and accumulation (Thompson & Silverman, 2008). Figure 3 schematically describes the connections between the main components of the problem.

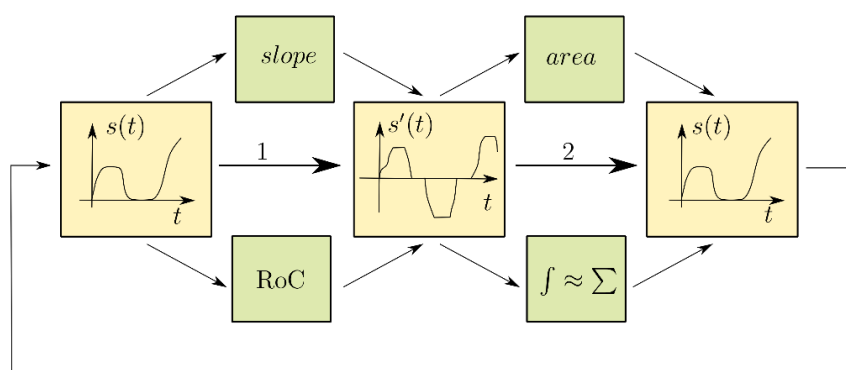


Figure 3: Structure of the problem

During the first session (marked by “1” in Figure 3), we expected the students to sketch the graph of the velocity using either the geometrical interpretation of the derivative as the slope of a tangent line or a numerical approximation of the RoC of  $s(t)$ . In the second session (marked by “2” in Figure 3), when reconstructing the distance from the graph of the instantaneous velocity  $s'(t)$ , we expected the



students to do this by estimating the area under the curve or using Riemann sums for the approximation of a definite integral.

The warm-up task is phrased entirely in mathematical language and is highly scaffolded, paving the way for the main problem by introducing a routine (numerical differentiation) that can be used to approach it. It contains many visual mediators: formulas as well as a table of values. In contrast to the warm-up task, the main problem uses an everyday setting and everyday language, and contains only one visual mediator, the position-time graph, which provides a partial mathematization of the problem. Still, although the questions are phrased in scientific language, the problem formulation contains few mathematical terms. As for the follow-up problem about reconstructing the distance given the velocity graph, it is also built around a single graph, and is phrased in everyday language.

Although all questions posed are closed, they do not suggest particular solution methods. Thus, we avoid reducing student agency, instead providing students with the opportunity to engage with various mathematical routines, both familiar and less familiar. We deliberately chose a setting where the function did not have an algebraic realisation, to emphasise the need for numerical techniques for handling even very simple real-world problems (Kaput, 1994). Moreover, numerical differentiation of discrete data is a topic that is not covered in the lectures, but which is of great relevance for engineering students. To answer the first two questions, students need to interpret the graph and extract the information needed. These are both routines that should be well-known to them from their previous studies. They then need to use this information, first for estimating the instantaneous velocity, which in turn can be done either numerically, building on the notion of RoC and using the numerical differentiation routine introduced in the warm-up task, or geometrically, by estimating the slope of the graph at the indicated times. Whichever method the students choose, it will require them to engage with routines less familiar to them. They are then asked to estimate average velocity, a routine familiar to them from upper secondary mathematics and physics. Student difficulty with RoC has been widely documented (Bressoud, Ghedamsi, Martinez-Luaces & Törner, 2016), and as Hauger (2000) points out, students often make sense of instantaneous RoC in terms of average RoC. Therefore “calculus teachers should capitalize on this knowledge to help students learn about instantaneous rate of change and the derivative” (p. 896). The task allows students to reflect on the relationship between average velocity and instantaneous velocity, in terms of the position-time graph. Since we expect both numerical and geometrical routines to be used for estimating the velocity, there is opportunity for discussion about the relation between these two methods. In the last part of the problem, students then need to use one of the routines for estimating instantaneous velocity in order to find the information needed to engage in a graph construction routine, namely constructing the velocity-time graph. The relationship between such graphs is another topic where student difficulty is well documented (e.g. Berry & Nyman, 2003). Moreover, constructing the velocity-time graph allows students to see the connection between the derivative as a value at a specific point (instantaneous velocity) and the derivative as a function (velocity as it varies over time). By numerically estimating the instantaneous velocity at several points and plotting them against time, students can see the functional character of the relationship, even though it is not given by a formula. Finally, interpreting this graph in the context of the original phrasing of the problem requires students to transition between different realisations of the function, from graphical to numerical and back.

The follow-up problem also allows students to engage in different types of routines. Again, the lack of an algebraic expression requires students to use numerical integration routines, either for estimating the area under the velocity curve or by using an approximation with Riemann sums. This problem also provides an opportunity for students to see how the Fundamental Theorem of Calculus connects RoC and accumulation (Thompson & Silverman, 2008), by realising that the accumulated position, that is, the distance, at time  $t$  is given by the integral of its RoC, that is, the velocity, from the starting point to  $t$ . By doing this for different values of  $t$ , the students can gain a sense of the integral as an accumulation function of the quantity whose RoC we know. This is something that students often struggle with, since they are not used to thinking of the upper limit as varying (ibid.). A related difficulty, pointed out by Thompson and Silverman (ibid.), is that the role of the variable of integration is often a mystery to students, but by considering Riemann sums this role can be made clearer. When students have to decide how to divide the interval of integration, the integration variable becomes experientially real to them. In fact, we considered asking the students to reconstruct the whole position-time-graph, but decided against it since it would have been too cumbersome without access to computational software. The problem also requires students to interpret the negative area under the velocity curve in terms of accumulated position, something that research has shown to be a challenge for students (Bressoud et al, 2016).

The intervention for which the problem analysed in this paper was designed took place during the spring semester 2019. Intervention data is currently collected and analysed.

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# **Raising Calculus to the Surface: Extending derivatives and concepts with multiple representations**

Aaron Wangberg<sup>1</sup> and Elizabeth Gire<sup>2</sup>

<sup>1</sup>Winona State University, United States; [awangberg@winona.edu](mailto:awangberg@winona.edu)

<sup>2</sup>Oregon State University, United States; [giree@oregonstate.edu](mailto:giree@oregonstate.edu)

## **Raising Calculus to the Surface**

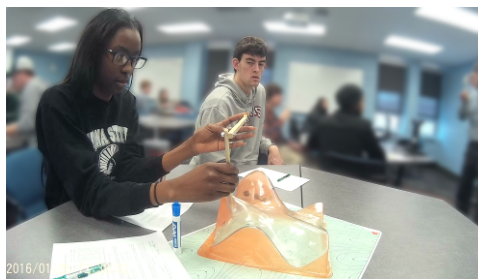
The Raising Calculus to the Surface project (RCS) uses small group activities and physical manipulatives to help students explore calculus concepts in the multivariable setting. By using dry erase markers and tools, like an inclinometer, to measure quantities on surface manipulatives and contour mats, as shown in Figure 1, students can engage in authentic mathematical practices, discuss new concepts, and uncover geometric relationships behind major theorems prior to formal instruction. Formulas for the surface functions are kept secret from students and instructors, providing opportunity for students to connect physical characteristics of the representations with the contextualized activities as they discuss and debate concepts even before mastering symbolic notations. Instructors at 65 institutions spanning high school, two-year and four-year institutions have adopted the materials. The project has been adopted by seven physicists, and a related project, Raising Physics to the Surface, is extending the approach for undergraduate physics courses.

## **Background**

The multiple external representations (MERs) created by the Raising Calculus project help students engage meaningfully with mathematical content. Ainsworth (2006) proposes a conceptual framework, named DeFT, for considering learning with MERs. DeFT considers the Design parameters that are unique to learning with MERs, the Functions that MERs serve in supporting learning, and the cognitive Tasks that must be undertaken by a learner interacting with MERs. Ainsworth identifies five design dimensions for multi-representational systems: number; distribution of information among representations; form; the sequence of representations; support for translation between representations. Ainsworth (1999) notes combining MERs allows a second representation to (a) support complementary processes or information contained within the first representation, (b) constrain the interpretation of the first representation, or (c) support the construction of deeper understanding when learners achieve insight using a second representation.

## **Data Collection**

Students enter multivariable calculus with various conceptions of derivative. Used prior to formal lecture in multivariable calculus, the RCS materials let students extend their prior conceptions. The following discussion is based upon the activity design, classroom video data, and the experiences of the instructor working in the classroom with 21 multivariable calculus students at a mid-sized regional university. Audio and video data captured seven groups of three students working with several RCS manipulatives during a one hour activity. This provides seven total hours of data. The video data, combined with the instructor's experience, are used to identify how the various MERs associated with the RCS activity help students explore new partial derivatives concepts.



**Figure 1: Students measure derivatives on a Surface with an inclinometer**

## **The activity and representations**

The *Hotplate* activity is designed to help students develop a deeper understanding of partial derivatives. In the activity, students work with various MERs including a *surface* (a tangible graph of a multivariable function) and *rectangular coordinate grid*, a *contour plot*, and an *inclinometer tool* used to measure slope. The activity sheet uses *symbols*  $\partial T/\partial x$  and  $\partial T/\partial y$  to represent partial derivatives and *words* to provide a context (e.g. *the surface represents the temperature of a hot plate*), but  $T$  is not explicitly defined as *temperature* nor is a formula provided for  $T$ .

During the activity, students (1) identifying locations on the surface matching conditions expressed symbolically (e.g.  $\partial T/\partial x < 0$  and  $\partial T/\partial x > 0$ ), (2) measure partial derivatives on the surface with the inclinometer, and (3) rank the value of a partial derivatives at three points marked on a contour plot.

## **Initial transfer between representations with different forms**

Students coordinate three MERs at the activity's start: *words* (for context) and *symbols* on the activity sheet, and a *surface with a rectangular coordinate grid*. These MERs contain similar information accessible through context: The activity sheet contains unfamiliar symbols<sup>1</sup>, like  $\partial T/\partial x$  or  $\partial T/\partial y$ , which never appear on any other MER. The *words* on the activity sheet, which say “the surface is the temperature of a 10 inch x 10 inch hotplate”, provide a situation that students can use to translate between representations.

The MER's forms help students transfer characteristics between them: The surface and contour grid are public, shared spaces used by three students, while the activity sheet containing the symbols and context are private spaces. Students often use language like “*temperature is increasing here*” or “*temperature is positive here*” when pointing to the surface and referring to “ $\partial T/\partial x > 0$ ” on the activity sheet. These sloppy descriptions for “ $\partial T/\partial x > 0$ ” cause confusion but encourage discussion surrounding the meaning of concepts available on the different representations and helps them see the necessity of using precise wording.

## **Representations constraining function of partial derivative**

Pre-test assessment data indicated students often describe derivatives as “rules” or “slope”, even though they may have other descriptions of derivative. By representing the temperature with a

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<sup>1</sup> Students may have seen  $dy/dx$  or  $dp/dt$  notations instead of  $f'$  notations for derivatives in prior calculus courses.

surface, and withholding the formula for it, the activity and MER selection intentionally restricted students from accessing derivative rules to discuss derivatives.

### **Constraining function to measure partial derivatives on a surface**

With the surface, students quickly recognize that direction is crucial for partial derivatives: Every direction has a different slope at a point on the surface. When sitting on opposite sides of a surface, students would disagree whether “ $\partial T/\partial x$  is negative” or “ $\partial T/\partial x$  is positive” at a point on the surface. They interpreted slope relative to their position instead of to the coordinate axes. Drawing the axes was not enough; Indicating the increasing  $x$  and  $y$  directions alleviated the problems.

The inclinometer tool measures rate of change along one direction, thus accessing slope. It requires measuring changes in the  $T$  and  $y$  directions to form  $\partial T/\partial y$ . Combining the surface and inclinometer MERs simplified the process of measuring  $\partial T/\partial y$  by constraining the problem to one direction.

### **Complementary tasks and information**

The second part of the activity asks students to measure  $\partial T/\partial y$  at a point where  $\partial T/\partial y < 0$ . Students often made gave their individual measurements positive signs and added a negative sign to their final ratio. When asked to explain why this quantity was negative, students often provided answers such as “*It is negative because the slope is negative*” or “*It is negative because the surface falls in the  $y$ -direction*”, keeping the sign disconnected from their measurements. Since the inclinometer’s configuration requires the tool be pointed toward the negative  $y$ -axis for this measurement, several students used this configuration to explain the negative sign for their slope. Only when students allowed the changes in  $T$  and  $y$  to have signs could they recognized the negative sign for  $\partial T/\partial y$  as describing a relative sign difference between the changes in the  $T$  and  $y$  directions.

### **Constructing derivatives with multiple representations**

*The Hotplate* concludes with students ranking  $\partial T/\partial x$  and  $\partial T/\partial y$  at points on a contour map. The dimensionality and form differences between the surface and contour plot MERs made it difficult for students to translate the “slope” notion of derivative to the contour map. The contour map collapses three dimensions of information onto two dimensions, flattening spatial notions of slope. Some students tried visualizing the contour data; visualizing and comparing slopes proved difficult.

Despite the different forms of the contour map and surface, the notion of derivative as a ratio of small changes nicely matched the construction of partial derivatives using the inclinometer. After drawing a vector on the contour map, students could interpret the change in temperature and change in the domain along this directed vector and form  $\partial T/\partial y$ . Students discussed whether these arrows should be unit length or extend from one known contour line to another to use known values of  $t$ .

### **Discussion**

Zandieh’s theoretical framework (2001) progresses through increasingly sophisticated notions of derivative to ratios of small changes in two quantities. Roundy et al. (2015) extend this with a Numerical representation and introduce the concept of a “thick derivative”, a notion explained in Dray (2016, to appear), as a notion used by scientists in situations, like experiments, that involve discrete data or factors making it impractical or impossible to utilize the Limit notion for derivative.

The MERs involved in *The Hotplate* certainly restricted students from using some notions of derivative, like derivative rules, but provided opportunity for students to discuss direction and interpretations of derivatives. The activity illustrates that notions of slope and direction were very accessible to students when working with the surface representation. Although the notion of slope was limiting when working with contour maps, the sequence of MERs in the activity supported re-interpretation of slope as a notion of ratio of small changes along a specified direction. This was a useful derivative notion applicable for the surface, contour representations, symbolic, and word representations. When working with the contour plot, students broached the subject of *thick derivatives*, and such a discussion could help translate derivative concepts to science experiments.

## Implications

Looking back to first-semester calculus, the *Hotplate* activity illustrates several opportunities for helping students discuss calculus concepts: First, combining explorations with (tangible) MERs (without formulaic expressions) enabled students to discuss mathematical concepts separate from symbolic computation. Second, asking students to find features described in context provided practice orienting and setting up problems. This skill, heavily used in science, likely impacts transfer. Third, emphasizing the differential quantities  $dT$  and  $dx$  are signed quantities to emphasize that the derivative is a ratio of small changes which is readily adaptable to multiple situations in mathematics as well as experimental settings involving measurement in science.

## Acknowledgment

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